

**VECTORS**

*A vector is a matrix with only 1 column.*

To enter a vector, use menu Insert / Matrix, or quicker, hit Ctrl+m. Use the tab key to go from coefficient to coefficient while entering the numbers:

$$\mathbf{v} := \begin{pmatrix} 1 \\ 10 \\ 100 \\ 1000 \\ 1000000 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 10 \\ 100 \\ 1 \times 10^3 \\ 1 \times 10^6 \end{pmatrix}$$

To change the format of the result, use menu Format / Result. Or more quickly, double click:

$$\mathbf{v} = \begin{pmatrix} 1 \\ 10 \\ 100 \\ 1000 \\ 1000000 \end{pmatrix}$$

This used the "decimal" format, (or "exponential treshold" 7).

$$\mathbf{v} = \begin{pmatrix} 1 \times 10^0 \\ 10 \times 10^0 \\ 100 \times 10^0 \\ 1 \times 10^3 \\ 1 \times 10^6 \end{pmatrix}$$

This used the "engineering" format.

## MATRICES AND SOLVING SYSTEMS OF LINEAR EQUATIONS

Consider the following system of equations for three unknowns  $x_1$ ,  $x_2$ , and  $x_3$ :

$$\begin{aligned} 2x_2 + 3x_3 &= 9 \\ 2x_1 + 3x_2 + 4x_3 &= 8 \\ 5x_1 + 6x_2 + 7x_3 &= 9 \end{aligned}$$

The coefficients multiplying the unknowns form a "matrix", (some table), call it **A**. The right hand sides form a right hand side vector, call it **b**. The unknowns form a vector of unknowns, call it **x**.

Note that the unknowns *must* be ordered and you must enter a zero for missing unknowns:

$$\mathbf{A} := \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad \mathbf{b} := \begin{pmatrix} 9 \\ 8 \\ 9 \end{pmatrix}$$

If the equations have units, make sure that all unknowns are scaled to the same units and all right hand sides are. Also scale so that these quantities are not very large or small.

You can let Mathcad find the solution for you. The **WRONG** way to do it is by pre-multiplying **b** by the inverse matrix after checking that the determinant is nonzero:

$$\text{Wrong:} \quad ||\mathbf{A}|| = 3 \quad \mathbf{x} := \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} -1.333 \\ -4 \\ 5.667 \end{pmatrix}$$

Until you become an expert in Numerical Linear Algebra, (or are told so), you should *never* compute a determinant or an inverse matrix. That is normally a very bad idea! The only reason it worked above is because the matrix is so small.

Instead, first check the condition number of the matrix:

$$\mathbf{cnd} := \mathbf{cond2}(\mathbf{A}) \quad \boxed{\mathbf{cnd} = 85.824}$$

This result implies that the relative error in **x** may be up to 86 times larger than the relative error in **b**. That is not bad, because **b** will have a very small relative error like  $10^{-16}$ . So we can proceed to solve the equations. You should normally use an "LU decomposition" algorithm to do so. Mathcad provides "Isolve":

$$\mathbf{x} := \mathbf{Isolve}(\mathbf{A}, \mathbf{b}) \quad \boxed{\mathbf{x} = \begin{pmatrix} -1.333 \\ -4 \\ 5.667 \end{pmatrix}}$$

Let's also try this for a singular matrix. The above matrix becomes singular if we replace the zero in it by a 1. Let's do so. Individual matrix elements can be selected as:

$$\mathbf{A}[\text{row\_number}, \text{column\_number}] \text{SPACE} \quad \textit{Remember that.}$$

The element of **A** that we want to change is in the first row and the first column, so we want to enter  $\mathbf{A}[1,1]$ SPACE. Unfortunately, if the variable ORIGIN is set to zero, the default, counting of rows and columns starts at 0, not 1. So to use normal counting, we *must* first set ORIGIN to 1:

$$\boxed{\mathbf{ORIGIN} := 1} \quad \text{You MUST set ORIGIN and MUST have a box around it!}$$

$$A_{1,1} := 1$$

Note that the [1,1SPACE shows up as subscripts!

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$$

$$b = \begin{pmatrix} 9 \\ 8 \\ 9 \end{pmatrix}$$

Note that the first coefficient of A is now 1 while the others are unchanged.

Warning: Please keep matrix indices apart from text subscripts. For example, do not confuse b[2SPACE:  $b_2$ , from b.2:  $b_2$ .

In the first case, the 2 means "the second element of vector  $\mathbf{b}$ ", in the second case the 2 is just part of the name  $b_2$ .

Now that we have made A singular, try solving the system again. The proper way to do that, that you *must* use in your work is to first define a couple of functions if you did not already:

Note: use Ctrl+= for equals signs in conditions:

`badmat(cnd) := if(cnd + 0.001 = cnd, "The matrix is bad! The solution will be inaccurate!", "OK")`

`singmat(cnd) := if(cnd + 0.25 = cnd, "The matrix is singular to machine precision!!", badmat(cnd))`

Then compute and show the condition number:

$$\text{cnd} := \text{cond2}(A) \quad \boxed{\text{cnd} = 2.774 \times 10^{16}}$$

That is very bad, because the relative error in  $\mathbf{x}$  will no longer be small! Use singmat now::

$$\boxed{\text{singmat}(\text{cnd}) = \text{"The matrix is singular to machine precision!!"}}$$

If we would stick our head out, and solve anyway, we would get it chopped off:

$$\underline{\underline{\mathbf{x}}} := \text{lsolve}(A, \mathbf{b}) \quad \mathbf{x} = \begin{pmatrix} -2.252 \times 10^{15} \\ 4.504 \times 10^{15} \\ -2.252 \times 10^{15} \end{pmatrix}$$

Note that those numbers are complete nonsense. The correct solution is infinite, not something like  $10^{15}$ . *Better not put a block around these numbers!*

## MATRIX MANIPULATIONS

Let's reset  $\mathbf{A}$  and recompute  $\mathbf{x}$ :  $\mathbf{A}_{1,1} := 0$

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 9 \\ 8 \\ 9 \end{pmatrix} \quad \mathbf{x} := \text{lsolve}(\mathbf{A}, \mathbf{b}) \quad \mathbf{x} = \begin{pmatrix} -1.333 \\ -4 \\ 5.667 \end{pmatrix}$$

If that is the right solution, then  $\mathbf{A} \mathbf{x}$  should be  $\mathbf{b}$ . It is indeed:

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} -1.333 \\ -4 \\ 5.667 \end{pmatrix} \quad \mathbf{A} \cdot \mathbf{x} = \begin{pmatrix} 9 \\ 8 \\ 9 \end{pmatrix}$$

Note however how the product  $\mathbf{A} \mathbf{x}$  is computed. We have here a  $3 \times 3$  (3 rows  $\times$  3 columns) matrix  $\mathbf{A}$  times a  $3 \times 1$  (3 rows  $\times$  1 column) matrix  $\mathbf{x}$  (i.e. a vector  $\mathbf{x}$ ).

The first element, 9, of  $\mathbf{b}$  is the dot product of the first row of  $\mathbf{A}$  times the column of  $\mathbf{x}$ :

$$0 \cdot \mathbf{x}_1 + 2 \cdot \mathbf{x}_2 + 3 \cdot \mathbf{x}_3 = 9 \quad \text{or written out:} \quad 0 \cdot \left(-\frac{4}{3}\right) + 2 \cdot (-4) + 3 \cdot \left(5 + \frac{2}{3}\right) = 9$$

Similarly the second element of  $\mathbf{b}$  is the dot product of the second row of  $\mathbf{A}$  times the column of  $\mathbf{x}$ . And the third element of  $\mathbf{b}$  is the dot product of the third row of  $\mathbf{A}$  times the column of  $\mathbf{x}$ .

*Matrix multiplication is always row-column.*

Next note that  $\mathbf{x} \mathbf{A}$  is *not* the same as  $\mathbf{A} \mathbf{x}$ :

*Matrix multiplication does not commute;  $\mathbf{AB}$  is not normally the same as  $\mathbf{BA}$*

In fact,  $\mathbf{x} \mathbf{A}$  cannot even be computed, because the rows of  $\mathbf{x}$  contain only one element, and the columns of  $\mathbf{A}$  contain 3. There is no way to take a dot product between them.

(Try it)

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} -1.333 \\ -4 \\ 5.667 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} -1.333 \\ -4 \\ 5.667 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$$

$\begin{matrix} \text{rows} \times \text{columns} & \text{rows} \times \text{columns} & \text{rows} \times \text{columns} & \text{rows} \times \text{columns} \\ 3 \times 3 & 3 \times 1 & 3 \times 1 & 3 \times 3 \\ \wedge \text{-----} \wedge & & \wedge \text{-----} \wedge & \end{matrix}$

OK BAD

*The "inner dimensions" of multiplied matrices must be equal.*

Alternatively, the row dimension (number of elements in a row) of the first factor must equal the column dimension (number of elements in a column) of the second factor.

We cannot even matrix-multiply  $\mathbf{x}$  by itself to get the square length of the vector:

$$\mathbf{x} = \begin{pmatrix} -1.333 \\ -4 \\ 5.667 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} -1.333 \\ -4 \\ 5.667 \end{pmatrix}$$

3 x 1

3 x 1

We can however matrix-multiply  $\mathbf{x}$  by itself if we turn the column in the first factor into a row:

The "transpose" of a matrix  $\mathbf{A}$ , called  $\mathbf{A}^T$ , has the columns of  $\mathbf{A}$  turned into rows, and vice versa.

Note: use the View / Toolbars / Matrix toolbar or Ctrl+1 to get the transpose:

$$\mathbf{x}^T = (-1.333 \quad -4 \quad 5.667) \quad \mathbf{x} = \begin{pmatrix} -1.333 \\ -4 \\ 5.667 \end{pmatrix} \quad \sqrt{\mathbf{x}^T \cdot \mathbf{x}} = 7.063 \quad |\mathbf{x}| = 7.063$$

$\begin{matrix} 1 \times 3 & & 3 \times 1 \\ \wedge & \text{-----} & \wedge \\ & \text{OK} & \end{matrix}$

As seen above,  $\mathbf{x}^T \mathbf{x}$  gives the square length of vector  $\mathbf{x}$ .

One more thing:

For complex vectors, you want to take the "Hermitian Transpose". That means that you want to take complex conjugate in addition to transpose

Let's see how that works for some example complex vector.

To enter  $i$ , the square root of  $-1$ , hit  $i$  immediately after a number (use 1 if needed):

$$\mathbf{y} := \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

To take complex conjugate, use " on the variable.

$$\mathbf{y}^T = (1 \quad i) \quad \mathbf{y} = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \mathbf{y}^T \mathbf{y} = 0 \quad \overline{(\mathbf{y}^T)} = (1 \quad -i) \quad \mathbf{y} = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \overline{(\mathbf{y}^T)} \mathbf{y} = 2$$

You see why the complex conjugate is needed: without it, the *nonzero* vector  $\mathbf{y}$  would have a *zero* length!

Let's try the Hermitian product with some other complex vector:

$$\mathbf{z} := \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \overline{(\mathbf{y}^T)} = (1 \quad -i) \quad \mathbf{z} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \overline{(\mathbf{y}^T)} \mathbf{z} = -2i$$

Now suppose that you multiply  $\mathbf{x}$  by itself without taking transpose. Since a matrix multiplication is then not possible, Mathcad assumes that you want the dot product:

$$\mathbf{x} \cdot \mathbf{x} = 49.889 \quad \sqrt{\mathbf{x} \cdot \mathbf{x}} = 7.063$$

So far, so good. How about complex vectors?

$$\mathbf{y} \cdot \mathbf{y} = 2 \quad \mathbf{y} \cdot \mathbf{z} = 2i$$

Note that the second product has the wrong sign. And that older versions of Mathcad would give the first product *also* wrongly, as zero.

*Do not rely on Mathcad to do Hermitian products properly.*

Returning to real vectors, while as noted we cannot pre-multiply (left-multiply)  $\mathbf{A}$  by  $\mathbf{x}$ , we can pre-multiply both  $\mathbf{A}$  and  $\mathbf{A}^T$  with  $\mathbf{x}^T$ :

$$\mathbf{x}^T = (-1.333 \quad -4 \quad 5.667) \quad \mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad \mathbf{x}^T \mathbf{A} = (20.333 \quad 19.333 \quad 19.667)$$

$1 \times 3$                        $3 \times 3$                        $1 \times 3$                        $3 \times 3$

*Not the transpose of  $\mathbf{A} \mathbf{x}$ !*

$$\mathbf{x}^T = (-1.333 \quad -4 \quad 5.667) \quad \mathbf{A}^T = \begin{pmatrix} 0 & 2 & 5 \\ 2 & 3 & 6 \\ 3 & 4 & 7 \end{pmatrix} \quad \mathbf{x}^T \mathbf{A}^T = (9 \quad 8 \quad 9)$$

$1 \times 3$                        $3 \times 3$

This illustrates that to transpose a product, you must invert the order of the factors in addition to taking the transpose of each:

$$\boxed{(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T}$$

To illustrate how these things work out if both  $\mathbf{A}$  and  $\mathbf{B}$  are matrices, let's define a  $3 \times 2$  matrix  $\mathbf{X}$  that contains  $\mathbf{x}$  and  $2\mathbf{x}$  inside

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad \mathbf{X} := \text{augment}(\mathbf{x}, 2\mathbf{x}) = \begin{pmatrix} -1.333 & -2.667 \\ -4 & -8 \\ 5.667 & 11.333 \end{pmatrix}$$

$$\mathbf{A} \cdot \mathbf{X} = \begin{pmatrix} 9 & 18 \\ 8 & 16 \\ 9 & 18 \end{pmatrix} \quad \text{Is that what you expected?} \quad \mathbf{X}^T \mathbf{A}^T = \begin{pmatrix} 9 & 8 & 9 \\ 18 & 16 & 18 \end{pmatrix}$$

### IDENTITY MATRICES

While you are never supposed to compute an inverse matrix in numerical work, what happens if you do anyway and multiply  $A$  by its inverse?

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad \mathbf{A}^{-1} = \begin{pmatrix} -1 & 1.333 & -0.333 \\ 2 & -5 & 2 \\ -1 & 3.333 & -1.333 \end{pmatrix}$$

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2.665 \times 10^{-15} & 7.105 \times 10^{-15} & 1 \end{pmatrix} \quad \mathbf{A}^{-1} \cdot \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1.776 \times 10^{-15} \\ 0 & 0 & 1 \end{pmatrix}$$

You see here another reason not to compute inverse matrices; round-off errors! Both of these results *should* have been equal to the 3 x 3 "identity matrix".

$$\mathbf{identity}(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

An identity matrix is the generalization of the number 1 to matrices. Multiplying by the unit matrix, if possible, does not do anything:

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad \mathbf{identity}(3) \mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad \mathbf{A} \cdot \mathbf{identity}(3) = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} -1.333 & -2.667 \\ -4 & -8 \\ 5.667 & 11.333 \end{pmatrix} \quad \mathbf{identity}(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{X} \cdot \mathbf{identity}(2) = \begin{pmatrix} -1.333 & -2.667 \\ -4 & -8 \\ 5.667 & 11.333 \end{pmatrix}$$

## EIGENVALUES AND EIGENVECTORS

Eigenvalues and eigenvectors are important in many areas of engineering. For example, in vibrations, the eigenvalues give the frequencies of vibration, and the eigenvectors the mode shapes. Mathcad provides various functions to get them.

In general, if for some vector  $\mathbf{v}$ ,  $\mathbf{A} \cdot \mathbf{v}$  equals  $\lambda \cdot \mathbf{v}$ , then  $\mathbf{v}$  is an eigenvector of  $\mathbf{A}$  and  $\lambda$  the corresponding eigenvalue.

Let's go back to the singular matrix, because then we know one eigenvalue, namely zero. If a matrix is singular, there is at least one vector that pre-multiplied by  $\mathbf{A}$  produces zero. So that is an eigenvector of eigenvalue zero.

$$\mathbf{A}_{1,1} := 1$$

What we get from Mathcad is:

$$\mathbf{eigenvals}(\mathbf{A}) = \begin{pmatrix} 12 \\ -1 \\ 0 \end{pmatrix} \quad \text{Yes, one of the three eigenvalues is indeed zero.}$$

$$\mathbf{eigenvec}(\mathbf{A}, 0) = \begin{pmatrix} -0.408 \\ 0.816 \\ -0.408 \end{pmatrix} \quad \mathbf{A} \cdot \mathbf{eigenvec}(\mathbf{A}, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{Told you so.}$$

$$\mathbf{E} := \mathbf{eigenvecs}(\mathbf{A}) = \begin{pmatrix} -0.31 & -0.667 & 0.408 \\ -0.443 & -0.333 & -0.816 \\ -0.841 & 0.667 & 0.408 \end{pmatrix}$$

The matrix in the right hand side contains all three eigenvectors, as columns. To get them out use the View / Toolbars / Matrix toolbar or Ctrl+6:

$$\mathbf{E}^{\langle 1 \rangle} = \begin{pmatrix} -0.31 \\ -0.443 \\ -0.841 \end{pmatrix} \quad \mathbf{E}^{\langle 2 \rangle} = \begin{pmatrix} -0.667 \\ -0.333 \\ 0.667 \end{pmatrix} \quad \mathbf{E}^{\langle 3 \rangle} = \begin{pmatrix} 0.408 \\ -0.816 \\ 0.408 \end{pmatrix}$$

Let's check these vectors,

$$\mathbf{A} \cdot \mathbf{E}^{\langle 1 \rangle} = \begin{pmatrix} -3.72 \\ -5.314 \\ -10.096 \end{pmatrix} \quad 12\mathbf{E}^{\langle 1 \rangle} = \begin{pmatrix} -3.72 \\ -5.314 \\ -10.096 \end{pmatrix} \quad \mathbf{A} \cdot \mathbf{E}^{\langle 2 \rangle} = \begin{pmatrix} 0.667 \\ 0.333 \\ -0.667 \end{pmatrix} \quad \mathbf{A} \cdot \mathbf{E}^{\langle 3 \rangle} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Yes, OK; 12 times the vector, -1 times the vector, and 0 times the vector.



## ROTATION MATRICES

If you want to rotate things around an axis, you can do it with matrix multiplication. First define a unit vector  $\mathbf{n}$  along the direction of the desired axis of rotation. The vector should point in the direction that a normal screw would move when rotated that way. Then form the rotation matrix  $\mathbf{R}$  as shown below.

As an example, let's assume that we want to rotate around the axis that makes equal angles with the  $x$ ,  $y$ , and  $z$  axes of our coordinate system. A unit vector in that direction is:

$$\mathbf{n} := \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad \mathbf{n} = \begin{pmatrix} 0.577 \\ 0.577 \\ 0.577 \end{pmatrix}$$

Next we want to define a matrix  $\mathbf{n\_x}$  so that  $\mathbf{n\_x} \mathbf{r}$  gives the vectorial product between vectors  $\mathbf{n}$  and  $\mathbf{r}$ :

$$\mathbf{n\_x} := \begin{pmatrix} 0 & -\mathbf{n}_3 & \mathbf{n}_2 \\ \mathbf{n}_3 & 0 & -\mathbf{n}_1 \\ -\mathbf{n}_2 & \mathbf{n}_1 & 0 \end{pmatrix} \quad \mathbf{n\_x} = \begin{pmatrix} 0 & -0.577 & 0.577 \\ 0.577 & 0 & -0.577 \\ -0.577 & 0.577 & 0 \end{pmatrix}$$

We also need to select an angle to rotate over. Let's choose 120 degrees:

$$\mathbf{alp} := 120^\circ \quad \mathbf{alp} = 2.094$$

Now we are ready to create the rotation matrix

$$\mathbf{R} := \mathbf{n} \cdot \mathbf{n}^T - \cos(\mathbf{alp}) \cdot \mathbf{n\_x} + \sin(\mathbf{alp}) \mathbf{n\_x} \quad \mathbf{R} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Let's try it out. If we rotate a point on the  $x$ -axis, like  $x=2$ ,  $y=0$ ,  $z=0$  over 120 degrees around the given axis, it should become a point on the  $y$  axis. If we rotate once more, it should get on the  $z$ -axis, and after a third time (total of 360 degrees) it should be back where it was on the  $x$ -axis.

$$\mathbf{r} := \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{R} \cdot \mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \quad \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \quad \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Rotation matrices are unitary; their transpose is also their inverse:

$$\mathbf{R}^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$