EML 4930/5061	**Analysis in M.E. II**	**12/7&9/16**
Closed book	*Van Dommelen*	*9:30-10:20 am*

Show all reasoning and intermediate results leading to your answer, or credit will be lost. One book of mathematical tables, such as Schaum's Mathematical Handbook, may be used, as well as a calculator and one handwritten letter-size single formula sheet.

1. (20%) Solve the following PDE and boundary condition for $u(x, y)$ in the first quarter plane using the method of characteristics:

$$\cosh^2(x)u_x + u_y = -2yu \qquad u(x, 0) = e^{-x} \qquad 0 \le x \quad 0 \le y$$

Clean up your answer! Very neatly draw a set of characteristics to fully cover the complete quarter plane. Shade the region in which the initial condition determines the solution.

2. (20%) Use D'Alembert to find the deflection $u(x, t)$ of a string with fixed ends:

$$u_{tt} = 2^2 u_{xx} \qquad u(0, t) = u(\pi, t) = 0$$

if the initial string deflection is

$$u(x, 0) = \cosh(x)$$

and the initial string velocity is zero. In a very neat graph, show the initial condition extended to all x that makes the boundary conditions automatic. Evaluate $u(1, 24)$ exactly, fully simplified. Make sure to explicitly list the value of each term in the expression for it. Exact values only. *Simplify fully.*

3. (20%) Use the Laplace transform method *only* to solve the following heat conduction problem with linearized radiation:

$$u_t = u_{xx} - u \quad u(x, 0) = x \quad u(0, t) = f(t)$$

where f is a given function. Clean up completely.

Use only the attached Laplace transform tables unless stated otherwise. Use only one table item in each step you take (except P2) and list it! Use convolution only where it is unavoidable. No funny (discontinuous) functions in your answers.

4. (40%) Use separation of variables to solve the following problem of acoustics in a pipe with a closed end and an end closed by a moving piston:

$$u_{tt} = u_{xx}$$

with the initial conditions

$$u(x, 0) = 0 \qquad u_t(x, 0) = 0$$

and boundary conditions:

$$u_x(0, t) = 0 \qquad u_x(\pi, t) = \pi t^2$$

Show all reasoning. Show exactly what problem you are solving using separation of variables. Fully explore *all* possible eigenfunctions.

At the end, write out the fully worked out and *fully simplified* solution completely, with all parameters in it clearly identified. The professor should be able to simply take your final expressions and put them in a computer program to plot the solution without having to find stuff elsewhere.

Properties of the Laplace Transform

Property	$f(t)$	$\widehat{f}(s)$
P1: Inversion	$\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} \widehat{f}(s)e^{st}\,\mathrm{d}s$	$\displaystyle\int_0^\infty f(t)e^{-st}\,\mathrm{d}t$
P2: Linearity	$C_1 f_1(t) + C_2 f_2(t)$	$C_1\widehat{f_1}(s) + C_2\widehat{f_2}(s)$
P3: Dilation	$f(\omega t)$	$\omega^{-1}\widehat{f}(s/\omega)$
P4: Differentiation	$f^{(n)}(t)$	$s^n\widehat{f}(s) - s^{n-1}f(0^+) - \ldots - f^{(n-1)}(0^+)$
P5: Differentiation	$t^n f(t)$	$(-1)^n\widehat{f}^{(n)}(s)$
P6: Shift	$H(t-\tau)f(t-\tau)$ $H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$	$e^{-\tau s}\widehat{f}(s)$
P7: Shift	$e^{\sigma t}f(t)$	$\widehat{f}(s-\sigma)$
P8: Convolution	$\displaystyle\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$ Do *not* write as $f * g$	$\widehat{f}(s)\widehat{g}(s)$

Special Laplace Transform Pairs

	$f(t)$	$\widehat{f}(s)$		$f(t)$	$\widehat{f}(s)$
S1:	1	$\dfrac{1}{s}$	**S8:**	$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
S2:	t^n	$\dfrac{n!}{s^{n+1}}$	**S9:**	$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
S3:	$e^{\sigma t}$	$\dfrac{1}{s-\sigma}$	**S10:**	$t\sin(\omega t)$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$
S4:	$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$	**S11:**	$t\cos(\omega t)$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
S5:	$\dfrac{1}{\sqrt{\pi t}}e^{-k^2/4t}$	$\dfrac{1}{\sqrt{s}}e^{-k\sqrt{s}}$	**S12:**	$\sinh(\omega t)$	$\dfrac{\omega}{s^2-\omega^2}$
S6:	$\dfrac{k}{\sqrt{4\pi t^3}}e^{-k^2/4t}$	$e^{-k\sqrt{s}}$	**S13:**	$\cosh(\omega t)$	$\dfrac{s}{s^2-\omega^2}$
S7:	$\operatorname{erfc}\left(k/2\sqrt{t}\right)$	$\dfrac{1}{s}e^{-k\sqrt{s}}$	**S14:**	$\delta(t-\tau)$	$e^{-\tau s}$

Table 1: Properties of the Laplace Transform. $(k, \tau, \omega > 0,\ n = 1, 2, \ldots)$