EML 4930/5061	**Analysis in M.E. II**	**12/8/15**
Closed book	*Van Dommelen*	*8:00-10:00 pm*

Show all reasoning and intermediate results leading to your answer, or credit will be lost. One book of mathematical tables, such as Schaum's Mathematical Handbook, may be used, as well as a calculator and one handwritten letter-size single formula sheet.

1. (20%) Solve the following PDE and boundary condition for $u(x, y)$ using the method of characteristics:

$$x^3u_x + 2x^2yu_y = 2 \qquad u = x^2 + 2y \text{ on } y = 1 - x^2 \qquad 0 \le x \quad 0 \le y$$

Clean up your answer! Very neatly draw a set of characteristics to fully cover the complete first quadrant. Shade any regions in which the initial condition does not fix the solution. Indicate any singular points.

2. (20%) Use D'Alembert to find the acoustics u in an organ pipe with open ends:

$$u_{tt} = 5^2u_{xx} \qquad u_x(0, t) = u_x(2, t) = 0$$

if the initial string deflection and velocity are

$$u(x, 0) = e^x \qquad u_t(x, 0) = \cos(\pi x)$$

In two very neat graphs, show the initial conditions extended to all x that make the boundary conditions automatic. Evaluate the exact solution $u(0.7, 3)$. Make sure to explicitly list the value of each term in the expression for it. Exact value only. *Simplify fully.*

3. (20%) Use the Laplace transform method to solve the following damped wave propagation problem:

$$u_{tt} + 2u_t + u = u_{xx} \quad u(x, 0) = u_t(x, 0) = 0 \quad u_x(0, t) = f(t)$$

where f is a given function. Clean up completely.

Use only the attached Laplace transform tables unless stated otherwise. Use only one table item in each step you take (except P2) and list it! Use convolution only where it is unavoidable. No funny (discontinuous) functions in your answers.

4. (40%) Use separation of variables to solve the following damped wave propagation problem:

$$u_{tt} + 2u_t + u = u_{xx}$$

with the initial conditions

$$u(x, 0) = 0 \qquad u_t(x, 0) = 0$$

and boundary conditions:

$$u(0, t) = 0 \qquad u_x(\tfrac{1}{2}\pi, t) = t$$

Show all reasoning. Show exactly what problem you are solving using separation of variables. Fully explore *all* possible eigenfunctions.

At the end, write out the fully worked out and *fully simplified* solution completely, with all parameters in it clearly identified. The professor should be able to simply take your final expressions and put them in a computer program to plot the solution without having to find stuff elsewhere.

Properties of the Laplace Transform		
Property	$f(t)$	$\widehat{f}(s)$
P1: Inversion	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\widehat{f}(s)e^{st}\,\mathrm{d}s$	$\int_0^\infty f(t)e^{-st}\,\mathrm{d}t$
P2: Linearity	$C_1f_1(t)+C_2f_2(t)$	$C_1\widehat{f_1}(s)+C_2\widehat{f_2}(s)$
P3: Dilation	$f(\omega t)$	$\omega^{-1}\widehat{f}(s/\omega)$
P4: Differentiation	$f^{(n)}(t)$	$s^n\widehat{f}(s)-s^{n-1}f(0^+)-\ldots-f^{(n-1)}(0^+)$
P5: Differentiation	$t^nf(t)$	$(-1)^n\widehat{f}^{(n)}(s)$
P6: Shift	$H(t-\tau)f(t-\tau)$ $H(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}$	$e^{-\tau s}\widehat{f}(s)$
P7: Shift	$e^{\sigma t}f(t)$	$\widehat{f}(s-\sigma)$
P8: Convolution	$\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$ Do *not* write as $f*g$	$\widehat{f}(s)\widehat{g}(s)$

Special Laplace Transform Pairs					
	$f(t)$	$\widehat{f}(s)$		$f(t)$	$\widehat{f}(s)$
S1:	1	$\frac{1}{s}$	**S8:**	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
S2:	t^n	$\frac{n!}{s^{n+1}}$	**S9:**	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
S3:	$e^{\sigma t}$	$\frac{1}{s-\sigma}$	**S10:**	$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
S4:	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	**S11:**	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
S5:	$\frac{1}{\sqrt{\pi t}}e^{-k^2/4t}$	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}$	**S12:**	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
S6:	$\frac{k}{\sqrt{4\pi t^3}}e^{-k^2/4t}$	$e^{-k\sqrt{s}}$	**S13:**	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
S7:	$\operatorname{erfc}\left(k/2\sqrt{t}\right)$	$\frac{1}{s}e^{-k\sqrt{s}}$	**S14:**	$\delta(t-\tau)$	$e^{-\tau s}$

Table 1: Properties of the Laplace Transform. $(k,\tau,\omega>0,\ n=1,2,\ldots)$