EML 4930/5061 | **Analysis in M.E. II** | **04/29/14**
Closed book | *Van Dommelen* | *12:30-2:30 pm*

Show all reasoning and intermediate results leading to your answer, or credit will be lost. One book of mathematical tables, such as Schaum's Mathematical Handbook, may be used, as well as a calculator and one handwritten letter-size single formula sheet.

1. (20%) Solve the following PDE and boundary condition for $u(x,t)$ using the method of characteristics:

$$u_t - x\tan(t)u_x = \frac{u}{t} \qquad u = \cos(\tfrac{1}{2}x + \tfrac{1}{2}t) \text{ on } x = t \qquad 0 \le t \le 2\pi \quad 0 \le x$$

Clean up your answer! Very neatly draw a set of characteristics to fully cover the complete time range, with 5 or 6 characteristics in each subrange. Shade the regions in which the initial condition does not fix the solution.

2. (20%) Use D'Alembert to find the deflection u of the of a string of length 2:

$$u_{tt} = a^2 u_{xx} \qquad a = 3 \qquad u(0,t) = u(2,t) = 0$$

if the initial string deflection and velocity are

$$u(x,0) = e^x \qquad u_t(x,0) = \sin(\pi x)$$

In two very neat graphs, show the initial conditions extended to all x that make the boundary conditions automatic. Evaluate the exact solution $u(0.7,3)$. Make sure to explicitly list the value of each term in the expression for it. Exact value only, fully simplified.

3. (20%) Use the Laplace transform method to solve the following problem with a stabilized forcing at the end:

$$u_{tt} = a^2 u_{xx} \quad u(x,0) = u_t(x,0) = 0 \quad u(0,t) - p u_x(0,t) = f(t)$$

where a and p are positive constants and f is a given function. Clean up completely. The argument of f in your answer should be a single variable. No funny singular functions in your answers.

Use only the attached Laplace transform tables unless stated otherwise. List the items in the tables used. Use only one table item in each step you take! Use convolution only where it is unavoidable.

4. (40%) Use separation of variables to solve the following heat conduction problem in a bar with linearized radiation:

$$u_t = \kappa u_{xx} - 2u$$

where the initial temperature is zero and the end temperatures are as shown:

$$u(x,0) = 0 \qquad u(0,t) = 0 \qquad u(1,t) = e^{-t}$$

Show all reasoning. Show exactly what problem you are solving using separation of variables. Fully explore *all* possible eigenfunctions.

At the end, write out the fully worked out and fully simplified solution completely, with all parameters in it clearly identified. The professor should be able to simply take your final expressions and put them in a computer program to plot the solution without having to find stuff elsewhere.

Properties of the Laplace Transform

Property	$f(t)$	$\widehat{f}(s)$
P1: Inversion	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widehat{f}(s)e^{st}\,\mathrm{d}s$	$\int_0^\infty f(t)e^{-st}\,\mathrm{d}t$
P2: Linearity	$C_1 f_1(t) + C_2 f_2(t)$	$C_1\widehat{f_1}(s) + C_2\widehat{f_2}(s)$
P3: Dilation	$f(\omega t)$	$\omega^{-1}\widehat{f}(s/\omega)$
P4: Differentiation	$f^{(n)}(t)$	$s^n\widehat{f}(s) - s^{n-1}f(0^+) - \ldots - f^{(n-1)}(0^+)$
P5: Differentiation	$t^n f(t)$	$(-1)^n\widehat{f}^{(n)}(s)$
P6: Shift	$H(t-\tau)f(t-\tau)$ $H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$	$e^{-\tau s}\widehat{f}(s)$
P7: Shift	$e^{\sigma t}f(t)$	$\widehat{f}(s-\sigma)$
P8: Convolution	$\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$ Do *not* write as $f * g$	$\widehat{f}(s)\widehat{g}(s)$

Special Laplace Transform Pairs

	$f(t)$	$\widehat{f}(s)$		$f(t)$	$\widehat{f}(s)$
S1:	1	$\frac{1}{s}$	**S8:**	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
S2:	t^n	$\frac{n!}{s^{n+1}}$	**S9:**	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
S3:	$e^{\sigma t}$	$\frac{1}{s-\sigma}$	**S10:**	$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
S4:	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	**S11:**	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
S5:	$\frac{1}{\sqrt{\pi t}}e^{-k^2/4t}$	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}$	**S12:**	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
S6:	$\frac{k}{\sqrt{4\pi t^3}}e^{-k^2/4t}$	$e^{-k\sqrt{s}}$	**S13:**	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
S7:	$\operatorname{erfc}\left(k/2\sqrt{t}\right)$	$\frac{1}{s}e^{-k\sqrt{s}}$	**S14:**	$\delta(t-\tau)$	$e^{-\tau s}$

Table 1: Properties of the Laplace Transform. $(k, \tau, \omega > 0,\ n = 1, 2, \ldots)$