EML 4930/5061	**Analysis in M.E. II**	**04/29/14**
Closed book	*Van Dommelen*	*3-5 pm*

Show all reasoning and intermediate results leading to your answer, or credit will be lost. One book of mathematical tables, such as Schaum's Mathematical Handbook, may be used, as well as a calculator and one handwritten letter-size single formula sheet.

1. (20%) Solve the following PDE and boundary condition for $u(x, y)$:

$$\frac{1}{y}u_x + \frac{1}{x}u_y = 1 \qquad u(x, 2) = \sin x$$

   Clean up your answer!

   Draw the characteristics very neatly. Using your picture, explain where in the x, y-plane the derived solution can reasonably be assumed to apply, and where it is indeterminate or likely not to be correct.

2. (20%) Use D'Alembert to find the pressure u of the acoustics in a pipe,

$$u_{tt} = a^2 u_{xx} \qquad a = 2$$

   with one end open and one end closed,

$$u(0, t) = 0 \qquad u_x(1, t) = 0$$

   triggered by

$$u(x, 0) = e^x \qquad u_t(x, 0) = \sin(\tfrac{1}{2}\pi x)$$

   Evaluate the exact solution $u(0.1, 4.1)$. Make sure to explicitly list the value of each term in the expression for it. Exact value only, fully simplified. Graphs must be very neat.

3. (20%) Use the Laplace transform method to solve the following linearized "convection-diffusion problem" in a semi-infinite pipe, in which the flow velocity at one end of the pipe is given:

$$u_t + 2u_x = u_{xx} \quad u(x, 0) = 0 \quad u(0, t) = f(t)$$

   Clean up completely.

   Use only the attached Laplace transform tables unless stated otherwise. List the items in the tables used. No convolution unless it is unavoidable.

4. (40%) Use separation of variables to solve the following problem for acoustics in a pipe,

$$u_{tt} = a^2 u_{xx} \qquad a = 2$$

   driven by a varying external pressure at the open end,

$$u(0, t) = t - \tfrac{1}{6}t^3 \qquad u_x(1, t) = 0$$

   given the initial conditions

$$u(x, 0) = 0 \qquad u_t(x, 0) = 2$$

   Show all reasoning. Show exactly what problem you are solving using separation of variables.

   At the end, write out the fully worked out solution completely, with all parameters in it clearly identified. List all eigenfunctions and eigenvalues and Fourier coefficients completely.

Properties of the Laplace Transform

Property	$f(t)$	$\widehat{f}(s)$
P1: Inversion	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widehat{f}(s)e^{st}\,\mathrm{d}s$	$\int_0^\infty f(t)e^{-st}\,\mathrm{d}t$
P2: Linearity	$C_1 f_1(t) + C_2 f_2(t)$	$C_1\widehat{f_1}(s) + C_2\widehat{f_2}(s)$
P3: Dilation	$f(\omega t)$	$\omega^{-1}\widehat{f}(s/\omega)$
P4: Differentiation	$f^{(n)}(t)$	$s^n\widehat{f}(s) - s^{n-1}f(0^+) - \ldots - f^{(n-1)}(0^+)$
P5: Differentiation	$t^n f(t)$	$(-1)^n\widehat{f}^{(n)}(s)$
P6: Shift	$H(t-\tau)f(t-\tau)$ $H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$	$e^{-\tau s}\widehat{f}(s)$
P7: Shift	$e^{\sigma t}f(t)$	$\widehat{f}(s-\sigma)$
P8: Convolution	$\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$ Do *not* write as $f * g$	$\widehat{f}(s)\widehat{g}(s)$

Special Laplace Transform Pairs

	$f(t)$	$\widehat{f}(s)$		$f(t)$	$\widehat{f}(s)$
S1:	1	$\frac{1}{s}$	**S8:**	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
S2:	t^n	$\frac{n!}{s^{n+1}}$	**S9:**	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
S3:	$e^{\sigma t}$	$\frac{1}{s-\sigma}$	**S10:**	$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
S4:	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	**S11:**	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
S5:	$\frac{1}{\sqrt{\pi t}}e^{-k^2/4t}$	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}$	**S12:**	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
S6:	$\frac{k}{\sqrt{4\pi t^3}}e^{-k^2/4t}$	$e^{-k\sqrt{s}}$	**S13:**	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
S7:	$\mathrm{erfc}\left(k/2\sqrt{t}\right)$	$\frac{1}{s}e^{-k\sqrt{s}}$	**S14:**	$\delta(t-\tau)$	$e^{-\tau s}$

Table 1: Properties of the Laplace Transform. $(k, \tau, \omega > 0,\ n = 1, 2, \ldots)$