EML 4930/5061 **Analysis in M.E. II** **04/27/10**
Closed book *Van Dommelen* *3-5 pm*

Show all reasoning and intermediate results leading to your answer, or credit will be lost. One book of mathematical tables, such as Schaum's Mathematical Handbook, may be used, as well as a calculator and one handwritten letter-size single formula sheet.

1. (20%) Solve the following PDE and boundary condition

$$2tu_t + xu_x = x \quad \text{for } -1 < x < 1,\ t \geq 0 \qquad u(1,t) = \frac{t^2}{1+t^2}$$

   Draw at least four characteristics very neatly in the xt-plane. Explain why there is no boundary condition at $x = -1$. Explain why there is no initial condition given.

2. (20%) Use D'Alembert to find the pressure u to the problem of acoustics in a pipe of length $\frac{1}{2}$ with one end closed,

$$u_{tt} = a^2 u_{xx} \qquad u_x(0,t) = u(\tfrac{1}{2},t) = 0$$

   if the initial conditions are

$$u(x,0) = \sin(\pi x) \quad u_t(x,0) = \cos(\pi x).$$

   Use your results to evaluate u at $x = 2.1$ and $t = 1.1$ if $a = 2$. Exact value only.

3. (40%) Solve the following wave propagation problem of acoustics in a pipe that is forced at its ends:

$$u_{tt} = a^2 u_{xx} \qquad u(x,0) = u_t(x,0) = 0 \quad u_x(0,t) = t^3 \quad u(\tfrac{1}{2}\pi,t) = \tfrac{1}{2}\pi t^3$$

   Show all reasoning. List the eigenfunctions and eigenvalues completely. Write out the fully worked-out solution precisely and completely. Hint: for brevity, write $\bar{q}_n$ as $t\bar{p}_n$.

4. (20%) Solve the following problem of supersonic flow along a wavy plate using the Laplace transform method:

$$u_{xx} - a^2 u_{yy} = 0 \qquad u(0,y) = u_x(0,y) = 0 \quad u_y(x,0) = f(x) \quad u(x,\infty) = 0$$

   where $f(x)$ is to be considered to be some given function and $x \geq 0, y \geq 0$. Show all steps and reasoning. Your answer should not have weird mathematics, but be in simple terms that anyone with a basic understanding of calculus can understand. Use convolution only if it is unavoidable. If $u_x(x,0)$ gives the pressure on the surface, then what is it in terms of $f(x)$?

Table 6.3: Properties of the Laplace Transform

$f(t)$	$\hat{f}(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$
1. $C_1 f_1(t) + C_2 f_2(t)$	$C_1\hat{f}_1(s) + C_2\hat{f}_2(s)$
2. $f(at)$	$a^{-1}\hat{f}(s/a) \qquad (a > 0)$
3. $f^{(n)}(t)$	$s^n\hat{f}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0) \qquad (n = 1, 2, \ldots)$
4. $t^n f(t)$	$(-1)^n\hat{f}^{(n)}(s) \qquad (n = 1, 2, \ldots)$
5. $e^{ct}f(t)$	$\hat{f}(s-c) \qquad (s = \text{const.})$
6. $H(t-b)f(t-b)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$	$e^{-bs}\hat{f}(s) \qquad (b > 0)$
7. $f * g(t) \equiv \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$	$\hat{f}(s)\hat{g}(s)$

Table 6.4: Laplace Transform Pairs

$f(t)$	$\hat{f}(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$
1. 1	$\frac{1}{s}$
2. t^n	$\frac{n!}{s^{n+1}} \qquad (n = 1, 2, \ldots)$
3. e^{kt}	$\frac{1}{s-k}$
4. $\sin(at)$	$\frac{a}{s^2+a^2}$
5. $\cos(at)$	$\frac{s}{s^2+a^2}$
6. $\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$
7. $\frac{1}{\sqrt{\pi t}}e^{-k^2/4t}$	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}} \qquad (k > 0)$
8. $\frac{k}{\sqrt{4\pi t^3}}e^{-k^2/4t}$	$e^{-k\sqrt{s}} \qquad (k > 0)$
9. $\operatorname{erfc}\left(k/2\sqrt{t}\right)$, where $\operatorname{erfc}(z) \equiv \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-u^2}\mathrm{d}u$	$\frac{1}{s}e^{-k\sqrt{s}} \qquad (k > 0)$