EML 4930/5930	**Analysis in M.E. II**	**04/28/09**
Closed book	*Van Dommelen*	*3-5 pm*

Show all reasoning and intermediate results leading to your answer, or credit will be lost. One book of mathematical tables, such as Schaum's Mathematical Handbook, may be used, as well as a calculator and one handwritten letter-size single formula sheet.

1. (20%) Solve the following PDE and initial condition

$$u_t + (1+x^2)u_x = -2xu \quad \text{for } -\infty < x < \infty,\ t \geq 0 \qquad u(x,0) = \sin x$$

   Draw at least four characteristics very neatly in the xt-plane. Based on the picture, is the above problem properly posed? Explain.

2. (20%) Use D'Alembert to find the pressure u to the problem of acoustics in a pipe of length π with both ends closed,

$$u_{tt} = a^2 u_{xx} \qquad u_x(0,t) = u_x(\pi,t) = 0$$

   if the initial conditions are

$$u(x,0) = x^2 \quad u_t(x,0) = \cos x.$$

   In particular evaluate u at $x = 10$ and $t = 5$ if $a = 2$. Exact value only. Note that the extension of $u_t(x,0)$ to all x will take a simple analytical form.

3. (40%) Solve the following problem of heat conduction in a bar of length 1 using separation of variables:

$$u_t = u_{xx} \qquad u(x,0) = 0 \quad u(0,t) = 0 \quad u(1,t) = \sin(\omega t).$$

   Write out the fully worked-out solution precisely and completely.

4. (20%) Solve the following problem of heat conduction plus radiation in a semi-infinite bar by Laplace transforming the problem as given:

$$u_t = u_{xx} - u \qquad u(x,0) = 0 \quad u_x(0,t) = f(t)$$

   where $f(t)$ is to be considered to be some given function and $x \geq 0, t \geq 0$. Show all steps and reasoning. Your answer should not have weird mathematics, but be in simple terms that anyone with a basic understanding of calculus can understand.

Table 6.3: Properties of the Laplace Transform

$f(t)$	$\hat{f}(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$
1. $C_1 f_1(t) + C_2 f_2(t)$	$C_1 \hat{f}_1(s) + C_2 \hat{f}_2(s)$
2. $f(at)$	$a^{-1}\hat{f}(s/a) \qquad (a > 0)$
3. $f^{(n)}(t)$	$s^n \hat{f}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0) \qquad (n = 1, 2, \ldots)$
4. $t^n f(t)$	$(-1)^n \hat{f}^{(n)}(s) \qquad (n = 1, 2, \ldots)$
5. $e^{ct} f(t)$	$\hat{f}(s - c) \qquad (s = \text{const.})$
6. $H(t-b)f(t-b)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$	$e^{-bs}\hat{f}(s) \qquad (b > 0)$
7. $f * g(t) \equiv \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$	$\hat{f}(s)\hat{g}(s)$

Table 6.4: Laplace Transform Pairs

$f(t)$	$\hat{f}(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$
1. 1	$\frac{1}{s}$
2. t^n	$\frac{n!}{s^{n+1}} \qquad (n = 1, 2, \ldots)$
3. e^{kt}	$\frac{1}{s-k}$
4. $\sin(at)$	$\frac{a}{s^2 + a^2}$
5. $\cos(at)$	$\frac{s}{s^2 + a^2}$
6. $\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$
7. $\frac{1}{\sqrt{\pi t}}e^{-k^2/4t}$	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}} \qquad (k > 0)$
8. $\frac{k}{\sqrt{4\pi t^3}}e^{-k^2/4t}$	$e^{-k\sqrt{s}} \qquad (k > 0)$
9. $\mathrm{erfc}\left(k/2\sqrt{t}\right)$, where $\mathrm{erfc}(z) \equiv \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-u^2}\mathrm{d}u$	$\frac{1}{s}e^{-k\sqrt{s}} \qquad (k > 0)$