EML 5060 | **Analysis in Mechanical Engineering I** | **12/10/19**
Closed book | *Van Dommelen* | *5:30-7:30 pm*

Solutions should be fully *derived* showing all intermediate results, using class procedures. *Show all reasoning.* Bare answers are absolutely not acceptable, because I will assume they come from your calculator (or the math handbook, sometimes,) instead of from you. You must state what result answers what part of the question. Answer exactly what is asked; you do not get any credit for making up your own questions and answering those. Use the stated procedures. Give exact, fully simplified, answers.

One book of mathematical tables, such as Schaum's Mathematical Handbook, may be used, as well as a calculator, and a handwritten letter-size formula sheet.

1. **Background:** Dynamical systems, even if nonlinear, are typically described by differential equations. First order scalar equations can often be solved exactly (if they have some simplifying feature).

   Question: As a simple example, solve the nonlinear ordinary differential equation

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\left(\frac{y}{x}\right) + \frac{y}{x}$$

   using the *class procedures* for this type of equation. Write y as a function of x. Neatly and accurately sketch the curve for $-1 \le x \le 1$ if $y(1) = \pi/2$, labeling the axes appropriately. (If there are any multiple valued functions in your solution, use the principal branch.)

2. **Background:** Vibrating systems are typically described by differential equations. For small amplitudes, these can often be solved exactly.

   Question: Using variation of parameters and the other class procedures, including showing clean up procedures, solve the forced vibrating system

   $$2\ddot{x} + 2x = \frac{1}{\cos t} \qquad x(0) = 1, \quad \dot{x}(0) = 2$$

3. **Background:** Nonlinear dynamical systems often cannot be solved analytically. However, valuable qualitative understanding can often be obtained using critical (stationary) point analysis.

   Question: Consider the following equation for a damped vibrating system with a spring that softens when extended:

   $$\ddot{x} + 2\dot{x} + 5\tanh x = 0$$

   Convert this equation in a first order system, then show that that system has only one stationary point, and that at the stationary point, the linearized problem takes the form

   $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

   Solve this system using the class procedures for systems. Then very accurately and neatly draw a typical solution line near the stationary point. Show the governing vectors. Make sure angles and slopes are right. Classify the stationary point. Could the stationary point analysis be qualitative wrong? Explain all.

Properties of the Laplace Transform

Property	$f(t)$	$\widehat{f}(s)$
P1: Inversion	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widehat{f}(s)e^{st}\,\mathrm{d}s$	$\int_0^\infty f(t)e^{-st}\,\mathrm{d}t$
P2: Linearity	$C_1 f_1(t) + C_2 f_2(t)$	$C_1\widehat{f_1}(s) + C_2\widehat{f_2}(s)$
P3: Dilation	$f(\omega t)$	$\omega^{-1}\widehat{f}(s/\omega)$
P4: Differentiation	$f^{(n)}(t)$	$s^n\widehat{f}(s) - s^{n-1}f(0^+) - \ldots - f^{(n-1)}(0^+)$
P5: Differentiation	$t^n f(t)$	$(-1)^n\widehat{f}^{(n)}(s)$
P6: Shift	$H(t-\tau)f(t-\tau)$ $H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$	$e^{-\tau s}\widehat{f}(s)$
P7: Shift	$e^{\sigma t}f(t)$	$\widehat{f}(s-\sigma)$
P8: Convolution	$\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$ Do *not* write as $f * g$	$\widehat{f}(s)\widehat{g}(s)$

Special Laplace Transform Pairs

	$f(t)$	$\widehat{f}(s)$		$f(t)$	$\widehat{f}(s)$
S1:	1	$\frac{1}{s}$	**S8:**	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
S2:	t^n	$\frac{n!}{s^{n+1}}$	**S9:**	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
S3:	$e^{\sigma t}$	$\frac{1}{s-\sigma}$	**S10:**	$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
S4:	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	**S11:**	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
S5:	$\frac{1}{\sqrt{\pi t}}e^{-k^2/4t}$	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}$	**S12:**	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
S6:	$\frac{k}{\sqrt{4\pi t^3}}e^{-k^2/4t}$	$e^{-k\sqrt{s}}$	**S13:**	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
S7:	$\operatorname{erfc}\left(k/2\sqrt{t}\right)$	$\frac{1}{s}e^{-k\sqrt{s}}$	**S14:**	$\delta(t-\tau)$	$e^{-\tau s}$

Table 1: Properties of the Laplace Transform. $(k, \tau, \omega > 0,\ n = 1, 2, \ldots)$