EML 5060 | **Analysis in Mechanical Engineering I** | **12/8/15**

Closed book | *Van Dommelen* | *5:30-7:30 pm*

Solutions should be fully *derived* showing all intermediate results, using class procedures. *Show all reasoning.* Bare answers are absolutely not acceptable, because I will assume they come from your calculator (or the math handbook, sometimes,) instead of from you. You must state what result answers what part of the question. Answer exactly what is asked; you do not get any credit for making up your own questions and answering those. Use the stated procedures. Give exact, fully simplified, answers.

One book of mathematical tables, such as Schaum's Mathematical Handbook, may be used, as well as a calculator, and a handwritten letter-size formula sheet.

1. **Background:** Differential equations of third and higher order show up, for example, in viscous flow and beam bending.

   Question: Solve the ordinary differential equation

   $$y''' - 2y'' - 3y' + 6y = 12 \qquad y(0) = 2 \quad y'(0) = 0 \quad y''(0) = 1$$

   with undetermined coefficients. You will need to solve a cubic equation. To do so, guess and verify the root that is an integer, then factor it out and solve the remaining quadratic.

2. **Background:** The Laplace transform is a primary way to study the stability and evolution of linearized dynamical systems, because it turns them into algebraic systems.

   Question: Use the Laplace transform to solve the following vibrating system that is perturbed by an impulse at time $t = 2$:

   $$\ddot{u} + 4\dot{u} + 13u = 3\delta(t-2) \qquad u(0) = 1 \quad \dot{u}(0) = 1$$

   A table of Laplace transforms is attached. Everything not in this table must be fully derived showing all reasoning. The convolution theorem may only be used where it is absolutely unavoidable. Do not use any complex numbers in your analysis (besides s.) You can only use *one* Laplace transform table entry at each step (except P2), and its table number must be listed. No funny (discontinuous) functions in your answer.

3. **Background:** While nonlinear systems of differential equations are usually not analytically solvable, their qualitative behavior can be understood by analyzing their critical points (and behavior at infinity).

   Question: Solve the following linearized stationary point problem using the class procedure for systems of ODE:

   $$\dot{x} = -3x - y \qquad \dot{y} = 13x + y$$

   In particular, find the solution line for the initial condition

   $$x(0) = 0 \quad y(0) = 6$$

   and accurately draw it in the x, y plane for all *positive* times. Make the plane big enough to show the behavior clearly.

Properties of the Laplace Transform

Property	$f(t)$	$\widehat{f}(s)$
P1: Inversion	$\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} \widehat{f}(s)e^{st}\,\mathrm{d}s$	$\displaystyle\int_0^\infty f(t)e^{-st}\,\mathrm{d}t$
P2: Linearity	$C_1 f_1(t) + C_2 f_2(t)$	$C_1\widehat{f_1}(s) + C_2\widehat{f_2}(s)$
P3: Dilation	$f(\omega t)$	$\omega^{-1}\widehat{f}(s/\omega)$
P4: Differentiation	$f^{(n)}(t)$	$s^n\widehat{f}(s) - s^{n-1}f(0^+) - \ldots - f^{(n-1)}(0^+)$
P5: Differentiation	$t^n f(t)$	$(-1)^n\widehat{f}^{(n)}(s)$
P6: Shift	$H(t-\tau)f(t-\tau)$ $H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$	$e^{-\tau s}\widehat{f}(s)$
P7: Shift	$e^{\sigma t}f(t)$	$\widehat{f}(s-\sigma)$
P8: Convolution	$\displaystyle\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$ Do *not* write as $f * g$	$\widehat{f}(s)\widehat{g}(s)$

Special Laplace Transform Pairs

	$f(t)$	$\widehat{f}(s)$		$f(t)$	$\widehat{f}(s)$
S1:	1	$\dfrac{1}{s}$	**S8:**	$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
S2:	t^n	$\dfrac{n!}{s^{n+1}}$	**S9:**	$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
S3:	$e^{\sigma t}$	$\dfrac{1}{s-\sigma}$	**S10:**	$t\sin(\omega t)$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$
S4:	$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$	**S11:**	$t\cos(\omega t)$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
S5:	$\dfrac{1}{\sqrt{\pi t}}e^{-k^2/4t}$	$\dfrac{1}{\sqrt{s}}e^{-k\sqrt{s}}$	**S12:**	$\sinh(\omega t)$	$\dfrac{\omega}{s^2-\omega^2}$
S6:	$\dfrac{k}{\sqrt{4\pi t^3}}e^{-k^2/4t}$	$e^{-k\sqrt{s}}$	**S13:**	$\cosh(\omega t)$	$\dfrac{s}{s^2-\omega^2}$
S7:	$\operatorname{erfc}\left(k/2\sqrt{t}\right)$	$\dfrac{1}{s}e^{-k\sqrt{s}}$	**S14:**	$\delta(t-\tau)$	$e^{-\tau s}$

Table 1: Properties of the Laplace Transform. $(k, \tau, \omega > 0,\ n = 1, 2, \ldots)$