PERGAMON

Chemical Engineering Science 56 (2001) 4339–4349

Chemical Engineering Science

www.elsevier.com/locate/ces

Robust controller design for multivariable nonlinear systems via multi-model H_2/H_∞ synthesis

Soumitri Kolavennu[a], Srinivas Palanki[b,*], Juan C. Cockburn[c]

[a] Honeywell Technology Center, Honeywell, Inc., 3660, Technology Drive, Minneapolis, MN 55418, USA
[b] Department of Chemical Engineering, Florida A&M University-Florida State University, College of Engineering, 2525 Pottsdamer street, Tallahassee, FL 32310-6046, USA
[c] Department of Electrical Engineering, Florida A&M University-Florida State University, Tallahassee, FL 32310-6046, USA

Received 10 December 1999; received in revised form 29 September 2000; accepted 12 December 2000

Abstract

Input/output (I/O) linearization via state feedback provides a convenient framework for designing controllers for multivariable nonlinear systems. However this approach does not account for parametric uncertainty which may lead to loss of stability and performance degradation. In this paper, a multi-model H_2/H_∞ approach is utilized to design a robust controller for minimum phase multivariable nonlinear systems that are subject to parametric uncertainty. It is first shown that a state feedback (inner loop) based on nominal parameters introduces nonlinear perturbations to the linear sub-system and results in loss of decoupling. The uncertain system is characterized in a form that provides a framework for robust controller design. Recent results from linear robust control theory are utilized to design an outer-loop controller to account for the parametric uncertainty. This methodology is illustrated via simulation of a regulation problem in a continuous stirred tank reactor.

Keywords: MIMO systems; Robust control; I/O linearization

1. Introduction

In the last two decades, there has been a significant effort in the development of the theoretical foundations of the differential geometric approach to design nonlinear controllers for multivariable nonlinear systems (Isidori, 1995; Nijmeijer & der Schaft, 1990). One of the main contributions of the differential geometric approach is input/output (I/O) linearization, which seeks to reduce the original nonlinear system to a linear one in an input–output sense, via state feedback (Singh & Rugh, 1972; Isidori & Ruberti, 1984). In the multivariable case, one seeks to design a feedback that reduces the system, from an input–output sense, to an aggregate of independent single input, single output (SISO) linear channels and is referred to as a noninteracting control or decoupling problem (Isidori, 1995). Once this is achieved, it is possible to impose any desired stable dynamics around each individual SISO linear system via pole placement. For instance, Daoutidis, Soroush, and Kravaris (1990) and Daoutidis and Kravaris (1991) applied this approach for control of chemical reactors. Enns, Bugajski, Hendrick, and Stein (1994) and Morton (1995) used this approach for the control of nonlinear aircraft models.

This design methodology assumes the availability of an accurate model for the nonlinear system to achieve I/O linearization. However, in most practical cases, the nonlinear model is only an *approximate* representation of the actual plant. In particular, the model is usually developed from scanty laboratory data and thus there is uncertainty in the model parameters (e.g. rate constants, heat transfer coefficients, mass transfer coefficients etc.). In this situation, the issue of how to design the outer loop becomes important because due to the parametric uncertainty, the I/O linearizing feedback based on the nominal model is (i) unable to cancel the nonlinearities exactly and (ii) is no longer an aggregate of independent SISO channels. The outer loop has to be designed in a robust manner to cope with this parametric uncertainty as well as the effect of interacting channels.

* Corresponding author. Tel.: +1-850-410-6163; fax: +1-850-410-6150.
E-mail address: palanki@eng.fsu.edu (S. Palanki).

0009-2509/01/$ - see front matter
PII: S0009-2509(01)00034-3

The issue of robust controller design in the I/O framework for nonlinear systems has been attracted attention recently. However results are available primarily for SISO systems; the controller design issues for MIMO systems in the face of parametric uncertainty are not well understood. Two different approaches have been followed in the literature for uncertain SISO systems. The first approach considers the effect of the uncertainty in the nonlinear model and use nonlinear techniques to account for the uncertainty. Most of these techniques are Lyapunov based (see for instance Kravaris & Palanki, 1988; Chou & Wu, 1995) and are applicable only when certain matching "conditions" are satisfied. The second approach considers the effect of uncertainty as a perturbation to the I/O linear model and use linear robust control techniques to account for this uncertainty. For instance, Christofides, Teel, and Daoutidis (1996), and Christofides and Daoutidis (1997) developed robust control strategies for a class of two time-scale systems where the perturbations affinely multiply the "fast" states. Kolavennu, Palanki, and Cockburn (2000), utilized a multi-model H_2/H_∞ approach to design a robust outer loop to account for inexact "linearization" due to parametric uncertainty in I/O linearizable SISO nonlinear systems. In this paper, we extend this multi-model approach to multivariable nonlinear systems with parametric uncertainty that are I/O linearizable. In Section 2, the multivariable robust nonlinear control problem is formulated and the basic concepts of I/O linearization of multivariable systems are reviewed. In Section 3, the effect of uncertainty on the diffeomorphism for I/O linearization is shown. The uncertain transformed system is characterized in a form that provides a framework for robust controller design. Recent results from linear robust control theory are utilized which account for, not only the parametric uncertainty, but also the effect of interacting terms. Robust stability for this controller is analyzed. In Section 4, this controller synthesis procedure is illustrated via a simulation example of a regulation problem in a multivariable continuous stirred tank reactor (CSTR). Finally, in Section 5, the major conclusions of this approach are discussed.

2. Problem formulation

Consider the following state-space model of a multi-input multi-output (MIMO) nonlinear system with parametric uncertainty

$$\dot{x} = f(x,\theta) + \sum_{i=1}^{m} g_i(x,\theta)u_i, \qquad y_j = h_j(x) \quad j = 1,\ldots,m, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of states, $u \in \mathbb{R}^m$ is the vector of manipulated inputs, $y \in \mathbb{R}^m$ the vector of measured outputs, and θ is a vector of uncertain parameters that takes values in a compact set $\Theta \subset \mathbb{R}^p$. The objective is to design a controller such that the closed loop system is stable and certain performance objectives, e.g., tracking, disturbance rejection, etc., are satisfied for all $\theta \in \Theta$.

A large class of uncertain nonlinear chemical processes, such as liquid phase reactors and distillation columns can be modeled as Eq. (1). Nonlinearities appear very often in chemical process models derived either from first principles (e.g. Arrhenius relation, radiation heat transfer) or empirically (e.g. power law models, Michaelis–Menten models). Furthermore, the process parameters (e.g. reaction rates, heat transfer coefficients) in the model are not exactly known; this introduces uncertainty in the model. Hence advanced process control algorithms are needed to design robust controller for these nonlinear systems. To design a robust controller for these systems, a multi-loop design approach is proposed. The inner loop uses state feedback to linearize the *nominal* process dynamics in the I/O sense. The outer loop controller is a robust controller that guarantees performance despite uncertainty in the model.

The following terms are reviewed from the literature Isidori, 1995) for MIMO systems. This review of standard results will be used in the next section to design a robust controller in the presence of parametric uncertainty.

Definition 1 (*Isidori, 1995*). A nonlinear MIMO system of the form (1) is said to have a relative degree r_i with respect to an output y_i if the vector

$$\begin{aligned} &L_g L_f^k h_i(x) \triangleq [L_{g_1} L_f^k h_i(x) \ldots L_{g_m} L_f^k h_i(x)] = \bar{\mathbf{0}}, \\ &\quad k = 0 \text{ to } r_i - 2, \\ &L_g L_f^k h_i(x) \triangleq [L_{g_1} L_f^k h_i(x) \ldots L_{g_m} L_f^k h_i(x)] \neq \bar{\mathbf{0}}, \\ &\quad k = r_i - 1. \end{aligned} \tag{2}$$

Essentially r_i is the smallest integer k for which the vector $L_g L_f^{k-1} h_i(x)$ has at least one non-zero component. In other words, at least one of the inputs u_j affects the output y_i after r_i integrations.

Definition 2 (*Isidori, 1995*). The characteristic matrix of the system is defined as

$$\beta(x) = L_g L_f^{r-1} h(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \ldots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \ldots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}_{m\times m}, \tag{3}$$

If a system represented by Eq. (1) has well-defined relative degree r_i for all outputs y_i with $r = \sum r_i$ ($r \leqslant n$) and the characteristic matrix (3) is full row rank, then there exists a diffeomorphism $(\eta, z) = T(x)$ given by

$$z^{(i)} = \begin{bmatrix} z_1^{(i)} \\ z_2^{(i)} \\ \vdots \\ z_{r_i}^{(i)} \end{bmatrix} = \begin{bmatrix} h_i(x) \\ L_f h_i(x) \\ \vdots \\ L_f^{r_i-1} h_i(x) \end{bmatrix}, \tag{4}$$

$$z = [z^{(1)}, z^{(2)}, \ldots, z^{(m)}]^{\mathrm{T}},$$

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-r} \end{bmatrix} = \begin{bmatrix} \phi_{r+1}(x) \\ \phi_{r+2}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix}, \tag{5}$$

where $\phi_{r+1}, \phi_{r+2} \cdots \phi_n$ are chosen such that

$$[\mathrm{d}z_1^{(1)}, \mathrm{d}z_2^{(1)}, \ldots, \mathrm{d}z_{r_1}^{(1)}, \ldots, \mathrm{d}z_1^{(m)}, \mathrm{d}z_2^{(m)}, \ldots, \mathrm{d}z_{r_m}^{(m)}, \ldots, \mathrm{d}\phi_{r+1}, \mathrm{d}\phi_{r+2} \ldots \mathrm{d}\phi_n]$$

are linearly independent. This transforms the system (1) to the normal form:

$$\dot{\eta} = q(z, \eta) + \sum_{j=1}^{m} p_j(z, \eta) u_j, \tag{6}$$

$$\begin{aligned} \dot{z}_1^{(i)} &= z_2^{(i)} \\ &\vdots \\ \dot{z}_{r_i-1}^{(i)} &= z_{r_i}^{(i)}, \\ \dot{z}_{r_i}^{(i)} &= \alpha_i(z, \eta) + \sum_{j=1}^{m} \beta_{ij}(z, \eta) u_j, \quad i = 1, \ldots, m, \end{aligned} \tag{7}$$

where $\alpha_i = L_f^{r_i} h_i(x)$ and β_{ij} is the (i, j)th entry in Eq. (3).

Eq. (7) represents m subsystems, each with r_i states, which form the linearizable part of Eq. (1). Once the system has been transformed to the above normal form a state feedback law can be designed to invert the nonlinearities that appear in the equations

$$\dot{z}_r = \alpha(z, \eta) + \beta(z, \eta) u, \tag{8}$$

where

$$\dot{z}_r = \begin{bmatrix} \dot{z}_{r_1}^{(1)} \\ \dot{z}_{r_2}^{(2)} \\ \vdots \\ \dot{z}_{r_m}^{(m)} \end{bmatrix} \quad \alpha = \begin{bmatrix} L_f^{r_1} h_1(x) \\ L_f^{r_2} h_2(x) \\ \vdots \\ L_f^{r_m} h_m(x) \end{bmatrix}$$

β is the characteristic matrix (3) and u is the input vector.

Consider an input–output linearizable, minimum phase MIMO nonlinear system, with relative degree r, that has a well defined normal form for all $\theta \in \Theta$. The above definitions indicate that there exists a diffeomorphism $(\eta, z) = T(x, \theta)$ which transforms system (1) into its normal form (Eqs. (6) and (7)). This diffeomorphism results in a linearizable subsystem if the value of θ is exactly known. However, since θ is uncertain, the diffeomorphism has to be based on some nominal value θ_o of θ. This results in inexact linearization and loss of decoupling. This could lead to loss of stability and/or performance degradation if a conventional I/O design is used.

To overcome this loss of stability and/or performance the dynamics of the system obtained by using a transformation based on the nominal model are studied and a robust controller design methodology for this system is derived in the next section. The following assumptions are made:

(1) The nonlinear process is modeled as Eq. (1) and f and g_i are smooth vector fields, and h is a smooth scalar field.
(2) Model uncertainty is assumed to be represented as uncertainty in the parameter vector θ
(3) The system represented by Eq. (1) is an input–output linearizable, minimum phase MIMO nonlinear system, with finite relative degree r, that has a well defined normal form for all $\theta \in \Theta$.
(4) The relative degree with respect to each output and stability properties of the original nonlinear system are not changed due to uncertainty.
(5) The characteristic matrix β represented by Eq. (3) has full row rank.

Assumption 1 is a technical assumption that ensures that the Lie algebra required for deriving the state feedback law can be generated. Assumption 2 states that only parametric uncertainty is considered and the issue of unmodeled dynamics is not addressed in this paper. This covers a fairly large class of chemical processes where the dominant kinetics are well known but the parameter estimates are poor. Assumptions 3 and 4 narrow down the applicability of the theoretical development to the class of systems that are input–output linearizable and minimum phase with well defined relative degree of r for all $\theta \in \Theta$. This assumption is satisfied in many practical systems with parametric uncertainty (see, for instance, Sampath, Palanki, & Cockburn, 1998; Sastry & Isidori, 1989). Since most nonlinear models for chemical processes are developed from material and energy balances, the model parameters (rate constants, heat and mass transfer coefficients etc.) are *constant*, though they may be uncertain. Consequently, in this paper, we do not consider the case where the model parameters are time-varying. The case where the parameters are time-varying can be handled by using a time-varying co-ordinate transformation (Palanki & Kravaris, 1997) or by a gain-scheduling approach (Sampath, Palanki, & Cockburn, 1999).

Assumption 5 ensures that the system is state controllable and a decoupling nonlinear control law exists. The case where the characteristic matrix is singular can be handled by choosing alternative outputs as shown in Soroush and Kravaris (1994). The assumption of finite relative degree and nonsingular characteristic matrix ensures that the input/output linearizing law results in decoupling in the absence of parametric uncertainty (Isidori, 1995).

3. Robust controller design

In this section a methodology for the robust controller design is outlined. First it is shown how system (1) is transformed by a diffeomorphism based on nominal parameters using Lemma 1. Then, the transformed uncertain system is characterized in a convenient, approximate linear form using Lemma 2. It is shown via Theorem 1 that this characterization provides a framework for robust controller design using multi-objective H_2/H_∞ synthesis. Finally, it is shown via Theorem 2 that this controller stabilizes the original nonlinear system.

Lemma 1. *The System* (1) *with an additive model for uncertainty*, *i.e. of the form*,

$$f(x,\theta) = f_o(x) + \delta f(x,\theta), \tag{9}$$

$$g_i(x,\theta) = g_{io}(x) + \delta g_i(x,\theta) \quad i = 1\ldots m, \tag{10}$$

under the nominal transformation $(\eta, z) = T(x, \theta_o)$ *and the nominal feedback law* $u(x) = \beta_o(x)^{-1}[-\alpha_o(x) + v]$ *renders the subsystem*:

$$\begin{aligned} \dot{z}_i^{(j)} &= z_{i+1}^{(j)} + \Delta_i^{(j)}, \quad 1 \leqslant i \leqslant r_j - 1, \quad 1 \leqslant j \leqslant m, \\ \dot{z}_{r_j}^{(j)} &= \Delta_{r_j}^{(j)} - \Delta_\beta^{(j)}\alpha_o + v_j + \sum_{k=1}^{m} (\Delta_{\beta k}^{(j)}) v_k, \quad 1 \leqslant j \leqslant m, \end{aligned} \tag{11}$$

where $\Delta_i^{(j)} = L_{\delta f} L_{f_o}^{i-1} h_j(x,\theta))$, $\Delta_\beta^{(j)} = L_{\delta g_j} L_{f_o}^{r_j - 1} h_j(x,\theta)\beta_o^{-1}$, $\Delta_{\beta k}^{(j)}$ *is the kth element of* $\Delta_\beta^{(j)}$ *and the subscript 'o' refers to the the system at* $\theta = \theta_o$.

Proof. For the system (1) with additive model for uncertainty, the nominal transformation $(\eta, z) = T(x, \theta_o)$ is given by

$$\eta_i = \phi_i(x), \quad 1 \leqslant i \leqslant n - r,$$

$$z_i^{(j)} = L_{f_o}^{i-1} h_j(x), \quad 1 \leqslant i \leqslant r_j, \quad 1 \leqslant j \leqslant m$$

In the new co-ordinates, system (1) can be written as

$$\begin{aligned} \dot{\eta}_i &= p_i(\eta, z) + \Delta_{\eta_i}(\eta, z, \theta) \\ &\quad + q_i(\eta, z)u + \tilde{\Delta}_{\eta_i}(\eta, z, \theta)u, \quad 1 \leqslant i \leqslant n - r, \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{z}_i^{(j)} &= z_{i+1}^{(j)} + \Delta_i^{(j)}(\eta, z, \theta), \quad 1 \leqslant i \leqslant r_j - 1, \\ &\quad 1 \leqslant j \leqslant m, \\ \dot{z}_{r_j}^{(j)} &= \alpha_{oj}(x) + \Delta_{r_j}^{(j)}(\eta, z, \theta) + [\beta_{oj}(x) + \Delta_{\beta 1}^{(j)}(\eta, x, \theta)]u, \\ &\quad 1 \leqslant j \leqslant m, \end{aligned} \tag{13}$$

where

$$\Delta_{\eta_i}(\eta, z, \theta) = L_{\delta f}\phi_i, \tag{14}$$

$$\Delta_{\eta_i}(\eta, z, \theta) = L_{\delta g}\phi_i, \tag{15}$$

$$\Delta_i^{(j)}(\eta, z, \theta) = L_{\delta f} L_{f_o}^{i-1} h_j, \tag{16}$$

$$\Delta_r^{(j)}(\eta, z, \theta) = L_{\delta f} L_{f_o}^{r-1} h_j, \tag{17}$$

$$\Delta_{\beta 1}^{(j)}(\eta, z, \theta) = L_{\delta g} L_{f_o}^{r-1} h_j, \tag{18}$$

$$\alpha_{oj}(\eta, z) = L_{f_o}^{r_j} h_j, \tag{19}$$

$$\beta_{oj}(\eta, z) = L_{g_{jo}} L_{f_o}^{r_j - 1} h_j, \tag{20}$$

β_{oj} is the jth row of the characteristic matrix. The inner-loop controller is chosen to cancel the nominal nonlinearities as

$$u(x) = \beta_o(x)^{-1}[-\alpha_o(x) + v], \tag{21}$$

which renders Eq. (13) equal to

$$\begin{aligned} \dot{z}_i^{(j)} &= z_{i+1}^{(j)} + \Delta_i^j, \quad 1 \leqslant i \leqslant r_j - 1, \quad 1 \leqslant j \leqslant m, \\ \dot{z}_{r_j}^{(j)} &= \Delta_{r_j}^{(j)} - \Delta_\beta^{(j)}\alpha_o + v_j + (\Delta_\beta^{(j)})v, \quad 1 \leqslant j \leqslant m, \end{aligned} \tag{22}$$

where $\Delta_\beta = \Delta_{\beta 1}\beta_o^{-1}$ and $\Delta_\beta^{(j)}$ is the jth row of Δ_β. This is same as Eq. (11). □

Remark 1. The uncertainty in θ induces two types of perturbations; one that acts directly on the integrators and one that acts on the control input v itself. Thus, v has to be designed for robustness with respect to both uncertainties.

Now, we characterize the uncertainty in a suitable manner to design an outer loop controller. I/O linearization uses coordinate transformation and state feedback to reduce the nonlinear system to a linear one. However, in the presence of uncertainties, this method does not give a perfectly linear model. Perturbations appear in the canonical form, as nonlinear functions of z, due to the presence of uncertainties. These perturbations are characterized so that linear robust control techniques can be applied in the outer loop. The perturbations acting on the chain of integrators are expanded using a Taylor series expansion around the steady state ($\mathbf{0}$) of the transformed states. This introduces a significant part (first-order terms) of the perturbations into the state matrix (which can be expressed as a polytopic family of systems). The higher order terms are not neglected they are characterized as bounded disturbances to the linear uncertain system. The perturbations acting on the input are also characterized so that the input matrix can also be expressed as a polytope. To achieve this the input perturbations are expressed as a Taylor series around the nominal perturbations (θ_o). Thus we can obtain a standard linear parameter dependent system (with bounded disturbance inputs), that can be expressed as a polytopic family. This procedure is elucidated mathematically using Lemma 2. It may be noted that this is different from the Jacobi linearization of the original nonlinear system. Only the perturbations arising due to uncertainties are linearized but not the whole model.

Lemma 2. *The subsystem of the form* (11) *can be characterized as the following standard linear subsystem*

$$\begin{aligned} \dot{z} &= A(\theta)z + B(\theta)v + d, \\ y &= Cz, \end{aligned} \tag{23}$$

where $A(\theta), B(\theta)$ are parameter dependent matrices, d is the vector of nonlinear perturbations represented as external bounded disturbances, v is the vector of inputs, y is the vector of outputs and z is the vector of states.

Proof. Define the following vectors of perturbations:

$$\Delta_\alpha = \begin{bmatrix} \Delta_1^{(1)} \\ \vdots \\ \Delta_{r_1-1}^{(1)} \\ \vdots \\ \Delta_1^{(m)} \\ \vdots \\ \Delta_{r_m-1}^{(m)} \end{bmatrix}, \tag{24}$$

$$\Delta_B = \begin{bmatrix} \Delta_{r_1}^{(1)} + \Delta_\beta^{(1)}\alpha_o \\ \Delta_{r_2}^{(2)} + \Delta_\beta^{(2)}\alpha_o \\ \vdots \\ \Delta_{r_m}^{(m)} + \Delta_\beta^{(m)}\alpha_o \end{bmatrix}, \tag{25}$$

$$\Delta_\beta = \begin{bmatrix} \Delta_\beta^{(1)} \\ \Delta_\beta^{(2)} \\ \vdots \\ \Delta_\beta^{(m)} \end{bmatrix}, \tag{26}$$

where Δ_{α_i} are the perturbations on the chain of integrators in all m subsystems, Δ_{B_i} are the perturbations in the last (r_ith) equation of the ith subsystem and Δ_{β_i} are the perturbations on the inputs in the ith subsystem. These perturbations arise due to the uncertainty in the model parameters. Since we assume that the uncertain parameters are bounded $\theta \in \Theta$ and that the system is minimum phase and has a well defined normal form for all θ, this implies that each of the above perturbations are bounded for all values θ. First, we characterize the perturbations Δ_α and Δ_B by formal Taylor series expansion around the steady state (**0**) as follows:

$$\Delta_{\alpha_i}(\eta, z, \theta) = \delta_{\alpha_i}(\theta)z + \tilde{\delta}_{\alpha i}(\eta, z, \theta), \quad 1 \leqslant i \leqslant r - m, \tag{27}$$

$$\Delta_{B_i}(\eta, z, \theta) = \delta_{B_i}(\theta)z + \tilde{\delta}_{Bi}(\eta, z, \theta), \quad 1 \leqslant i \leqslant m, \tag{28}$$

where δ_{α_i} and δ_{B_i} are row vectors linear in z, which will be introduced into the state matrix $A(\theta)$. The higher order terms $\tilde{\delta}_{\alpha_i}$ and $\tilde{\delta}_{B_i}$ will be grouped as a vector of nonlinear perturbations, Δ_A, and characterized as external disturbances.

In addition to the uncertainty in the dynamic equations representing the chain of integrators, there is also uncertainty, $\Delta_\beta(\theta, z, \eta)$, that acts directly on the input. This uncertainty is characterized so that the coefficients of the inputs are affine functions of the uncertain parameters. By Taylor series expansion around the nominal parameter set and steady states we can obtain

$$\Delta_{\beta_k}^{(m)}(\eta, \theta, z) = \delta_{\beta_k}^{(m)}(\theta) + \tilde{\delta}_\beta^{(m)}(\eta, z, \theta_0). \tag{29}$$

To negate the effects of $\tilde{\delta}_\beta$, one can use a prefilter for the input v (Sampath et al., 1998).

Then the system can be written as:

$$\begin{aligned} \dot{z} &= A(\theta)z + \Delta_A + \sum_{i=1}^{m} b_i(\theta)v_i, \\ y &= Cz, \end{aligned} \tag{30}$$

where

$$C = \begin{bmatrix} C_1 & 0 & \dots & 0 \\ 0 & C_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_m \end{bmatrix}. \tag{31}$$

C_i is a row vector of length r_i whose first element is 1 and the rest all are zeros. matrix whose $(r_i, 1)$ element is 1 and the rest are zeros. The vector $b_i(\theta)$ is defined as:

$$\begin{aligned} b_i(\theta) = {} & [\tilde{C}_1\delta_{\beta_i}^{(1)} \ \tilde{C}_2\delta_{\beta_i}^{(2)} \ \dots \ \tilde{C}_m\delta_{\beta_i}^{(m)}]^{\mathrm{T}} \\ & + e_k \quad \left(k = \sum_1^i r_j\right), \end{aligned} \tag{32}$$

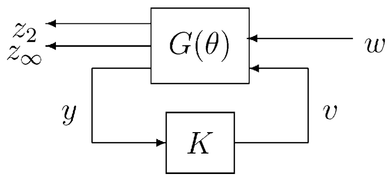

Fig. 1. Multi-model H_2/H_∞ synthesis problem.

where $\tilde{C}_i = C_i\tilde{I}_{r_i}$, $\tilde{I}_{r_i}$ is $r_i \times r_i$ square matrix similar to the identity matrix but the ones appearing on the reverse diagonal and e_k is the kth basis vector in $\mathbb{R}^r$. Define a vector of nonlinear perturbations, Δ_A, as follows:

$$\Delta_A = (\tilde{\delta}_{\alpha_1}, \ldots, \tilde{\delta}_{\alpha_{r-m}}, \tilde{\delta}_{B_1}, \ldots, \tilde{\delta}_{B_m})^{\mathrm{T}}. \tag{33}$$

This vector contains the higher order terms of $\tilde{\delta}_{\alpha_i}$ and $\tilde{\delta}_{B_i}$. This vector of nonlinear perturbations is represented as external bounded disturbances $d \in \mathscr{D}$. Recall that the perturbations are bounded for all θ. Let $\tilde{d}_i \in \mathscr{L}_2[0,\infty)$, such that $\|\tilde{d}_i\|_2 \leqslant 1$, $1 \leqslant i \leqslant r$. Weights W_{d_i} are chosen so that they capture the magnitude of the disturbance

$$\|\Delta_{A_i}\|_2 \leqslant \|W_{d_i}\tilde{d}_i\|_2, \quad 1 \leqslant i \leqslant r. \tag{34}$$

Then the effects of Δ_A can be represented by $W_d\tilde{d}$, where $W_d = \mathrm{diag}(W_{d_1}, \ldots, W_{d_r})$. This reduces Eq. (30) to

$$\dot{z} = A(\theta)z + \sum_{i=1}^{m} b_i(\theta)v_i + W_d\tilde{d}, \quad y = Cz \tag{35}$$

which is a particular form of the system represented by Eq. (23) with $B(\theta) = [b_1\ b_2\ \ldots\ b_m]$ and $d = W_d\tilde{d}$. □

To complete the design we must find a robustly stabilizing controller for the uncertain system (35). This is a linear robust control problem that can be solved via multi-objective optimization techniques such as H_2/H_∞ synthesis. The advantages of using this approach are (1) multiple performance (H_2) and robustness (H_∞) objectives can be placed in different channels of the MIMO system, (2) constraints on pole placement can be specified, and (3) a Lyapunov function for stability analysis can be explicitly constructed from the solution of the optimization problem. Alternatively, one could utilize a μ synthesis approach as shown in Sampath et al. (1998); however, H_2 performance objectives cannot be considered.

The mixed H_2/H_∞ synthesis with pole placement constraints technique can be used for robust design when the state space realization of the plant is affine in θ. The multi-model H_2/H_∞ state feedback synthesis places the poles such that the system has robust performance. This problem is represented in Fig. 1. The term, w, contains all external disturbances, e.g. d, and Z_2 and Z_∞ contain the relevant errors signals that we want to maintain small with respect to the 2-norm (average) and ∞-norm (worst case), respectively. The generalized plant $G(\theta)$ represents the plant model together with performance and normalization weights. The objective is to find a stabilizing controller K such that

$$a\|T_{Z_\infty w}\|_\infty^2 + b\|T_{Z_2 w}\|_2^2 \tag{36}$$

is minimized, for all $\theta \in \Theta$, where $T_{Z_\infty w}$ and $T_{Z_2 w}$ are linear operators mapping w to Z_∞ and w to Z_2 respectively and a, b are positive numbers representing the trade-off between the H_2/H_∞ objectives.

For the problem to be tractable, G should be affine in θ. If the matrix G is not affine, it poses a nonconvex, infinite dimensional optimization problem. For this reason, the uncertain state space model (35) is represented as a polytopic family of systems where the state space matrices are affine functions of the uncertain parameters i.e. of the form

$$A(\theta) = A_0 + \theta_1 A_1 + \cdots + \theta_k A_k + \cdots + \theta_p A_p \tag{37}$$

where, p is number of uncertain parameters. Then the multi-objective problem (36) is solved by linear matrix inequalities (LMI) using the following theorem.

Theorem 1 (*Khargonekar & Rotea, 1991*). *Consider a polytopic family of LTI systems of the form*

$$\dot{z} = A(\theta)z + W_d\tilde{d} + \sum_{i}^{m} b_i(\theta)v_i, \tag{38}$$

$$Z_\infty = C_1 z + D_{11}\tilde{d} + D_{12}v, \tag{39}$$

$$Z_2 = C_2 z + D_{22}v. \tag{40}$$

The state feedback that robustly stabilizes the above system and minimizes the performance objective (36) *is given by* $v = Kz$ *where* $K = YX^{-1}$ *and* X *and* Y *are obtained by solving the following optimization problem*:

$$\min_{Y,X,Q,\gamma^2} \quad a\gamma^2 + b\,\mathrm{Trace}(Q), \tag{41}$$

s.t.

$$\begin{pmatrix} A_kX + XA_k^{\mathrm{T}} + B_kY + Y^{\mathrm{T}}B_k^{\mathrm{T}} & W_d & XC_1^{\mathrm{T}} + Y^{\mathrm{T}}D_{12}^{\mathrm{T}} \\ W_d^{\mathrm{T}} & -I & D_{11}^{\mathrm{T}} \\ C_1X + D_{12}Y & D_{11} & -\gamma^2 I \end{pmatrix} < 0, \tag{42}$$

$$\begin{pmatrix} Q & C_2X + D_{22}Y \\ XC_2^{\mathrm{T}} + Y^{\mathrm{T}}D_{22}^{\mathrm{T}} & X \end{pmatrix} > 0, \tag{43}$$

$$\mathrm{Trace}(Q) < v_0^2 \tag{44}$$

$$\gamma < \gamma_0^2 \tag{45}$$

$$f_{\mathscr{D}}(X, Y) < 0, \tag{46}$$

for $k = 1, 2, \ldots$ (*number of parameters*) *where* $B = [b_1 \cdots b_2 \cdots b_m]$, A_k, B_k *are coefficients in the polytopic representation* (*as shown in Eq.* (37)) *of the parameter dependent state matrices A and B, respectively, and* γ_0 *and* v_0 *are upper bounds on the* H_∞ *and* H_2 *norms, respectively and* $f_{\mathscr{D}}(X, Y)$ *specifies the pole placement constraints.*

The minimization problem posed by Theorem 1 can be solved using standard software such as the LMI control toolbox in MATLAB (Gahinet, Nemirovski, Lamb, & Chilali, 1995).

Remark 2. Due to the presence of uncertainties, I/O decoupling is lost in the outer loop. Hence all the external reference inputs v_i affect each of the outputs y_i. The robust controller design takes this into account in the outer loop. A conventional I/O controller design assumes perfect decoupling and proceeds with a series of SISO outer loop controllers. This may lead to loss of stability and performance degradation in the presence of parametric uncertainty.

Remark 3. If a linear controller K cannot be found by solving the optimization problem (36) in Theorem 1, this does not imply that a robustly stabilizing controller does not exist for the original uncertain nonlinear system. This situation can arise when a bound on $\tilde{\delta}$ cannot be established or when the bound on $\tilde{\delta}$ is so large that the performance level γ cannot be satisfied for the uncertainty.

Now, it is shown that the feedback controller found by multi-objective synthesis robustly stabilizes the original nonlinear system.

Theorem 2. *Consider a nonlinear, uncertain system*

$$\dot{x} = f(x, \theta) + \sum_{i=1}^{m} g_i(x, \theta) u_i,$$
$$y_j = h_j(x), \quad j = 1, \ldots, m, \tag{47}$$

which can be represented, using Lemma 1, *as*

$$\dot{\eta}_i = p_i(\eta, z) + \Delta_{\eta_i}(\eta, z, \theta) + q_i(\eta, z)u + \tilde{\Delta}_{\eta_i}(\eta, z, \theta)u, \quad 1 \leqslant i \leqslant n - r, \tag{48}$$

$$\dot{z}_i^{(j)} = z_{i+1}^{(j)} + \Delta_i^{(j)}, \quad 1 \leqslant i \leqslant r_j - 1, \quad 1 \leqslant j \leqslant m,$$

$$\dot{z}_{r_j}^{(j)} = \Delta_{r_j}^{(j)} - \Delta_\beta^{(j)} \alpha_o + v_j + \sum_{k=1}^{m} (\Delta_{\beta k}^{(j)}) v_k \quad 1 \leqslant j \leqslant m,$$
$$y = Cz. \tag{49}$$

Now consider the parametric form of Eq. (49) *using Lemma* 2

$$\dot{z} = A(\theta) z + \sum_{i=1}^{m} b_i(\theta) v_i + W_d \tilde{d},$$
$$y = Cz. \tag{50}$$

Assume that θ *is in a compact set*, $\|\tilde{d}_i\|_2 \leqslant 1$, *and* $\|W_d\|_\infty$ *is finite. Then, a controller K that robustly stabilizes the system* (50) *also robustly stabilizes the nonlinear system* (47).

Proof. According to Lemma 1, characterization of the nonlinear perturbations that appear in Eq. (22) introduces the linear part of the Jacobi linearized perturbations into the state matrix, A and higher order terms as disturbances. Since it is assumed that the zero dynamics are stable for all values of θ, to prove stability of the nonlinear system, it is sufficient to prove that the uncertain subsystem (49) is stable. If one can find a Linear Robust Controller K that stabilizes this uncertain system in presence of the disturbances, then it stabilizes the subsystem (11). To see this consider a Lyapunov function for the linear system (35).

$$V = z^{\mathrm{T}} z \tag{51}$$

If $v = -Kz$ is chosen such that the system (35) is robustly stable, then

$$\dot{V} = z^{\mathrm{T}}(A(\theta) + A^{\mathrm{T}}(\theta))z + \sum_{i}^{m} b_i(\theta) v_i + 2z^{\mathrm{T}} W_d d \leqslant 0. \tag{52}$$

Now consider the nonlinear system (11) and write as,

$$\dot{z} = (F + \delta(\theta))z + \sum_{i}^{m} b_i(\theta) v_i + \tilde{\delta}(\eta, z, \theta), \tag{53}$$

where

$$F = \begin{bmatrix} F_1 & 0 & \ldots & 0 \\ 0 & F_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & F_p \end{bmatrix}, \tag{54}$$

$$F_i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{r_i \times r} \tag{55}$$

and $\tilde{\delta}(\eta, z, \theta) = [\tilde{\delta}_1, \ldots, \tilde{\delta}_r]^{\mathrm{T}}$, $\delta(\theta) = [\delta_1, \ldots, \delta_r]^{\mathrm{T}}$. The vector $\delta(\theta)$ captures the linear part of Eq. (27) and Eq. (28) and the vector $\tilde{\delta}(\theta)$ includes the higher order terms in these equations.

A Lyapunov function for this system is

$$V_n = z^{\mathrm{T}} z. \tag{56}$$

Thus

$$\dot{V}_n = z^{\mathrm{T}}(F + \delta(\theta) + F^{\mathrm{T}} + \delta^{\mathrm{T}}(\theta))z + \sum_{i}^{m} b_i v_i + 2z^{\mathrm{T}} \tilde{\delta}(\eta, z, \theta), \tag{57}$$

Given that $\tilde{\delta}(\eta, z, \theta) \leqslant \|W_d d\|$ it follows that

$$\dot{V}_n \leqslant \dot{V} \leqslant 0. \tag{58}$$

Hence the controller K stabilizes Eq. (11). This ensures that K stabilizes the original system (1). □

4. Illustrative example

Consider a continuous stirred tank reactor (CSTR) in which an isothermal, liquid phase, multi-component series-parallel chemical reaction is being carried out. The chemical reaction system is:

$$\begin{aligned} &A \to B \to C, \\ &A \to D. \end{aligned} \tag{59}$$

with the rates of reaction given by

$$r_A = k_1(T)C_A + k_3(T)C_A^2, \tag{60}$$

$$r_B = k_2(T)C_B - k_1(T)C_A, \tag{61}$$

where C_A, and C_B represent the concentrations of species A and B, respectively, and T represents the reactor temperature. The objective is to keep the concentration, C_B, and the temperature, T, at a desired set-point by manipulating the molar feedrate F/V and the heat input Q to the reactor. This system can be modeled as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} C_A \\ C_B \\ T \end{bmatrix} = \begin{bmatrix} -k_1(T)C_A - k_3(T)C_A^2 \\ k_1(T)C_A - k_2(T)C_B \\ \dfrac{1}{\rho C_p}(\Delta H_1 k_1(T)C_A + \Delta H_2 k_2(T)C_B \\ +\Delta H_3 k_3(T)C_A^2) \end{bmatrix} + \begin{bmatrix} C_{A0} - C_A & 0 \\ -C_B & 0 \\ \dfrac{T_0 - T}{\rho C_p} & \dfrac{1}{V\rho C_p} \end{bmatrix} \begin{bmatrix} F/V \\ Q \end{bmatrix}, \tag{62}$$

$$y = \begin{bmatrix} C_B \\ T \end{bmatrix}. \tag{63}$$

The reaction rate constants are dependent on the temperature as $k_i = A_i \exp(-E_i/RT)$. The numerical values of the system parameters used in this study are given in Table 1. There is uncertainty in the parameter A_2, it has a nominal value of 1.2×10^6 and may vary between 1.8×10^5 and 2.2×10^6. Thus, the objective is to design a controller that is robust to this parametric uncertainty.

The system is already in the normal form (Eqs. (6) and (7)) and both outputs have well defined relative degree of 1. Defining $\eta = C_A - C_{AS}$, $z_1 = C_B - C_{BS}$, $z_2 = T - T_S$,

Table 1
Values of the variables in illustrative example

Symbol	Value
$-\Delta H_1$	$4.5e5$ KJ
$-\Delta H_2$	$5.0e5$ KJ
$-\Delta H_3$	$6.0e5$ KJ
A_1	$2.0e6$ min^{-1}
A_2	$1.2e6$ min^{-1}
A_3	$1.2e6$ Kmol^{-1} min^{-1}
E_1	$5.0e4$ KJ
E_2	$6.5e4$ KJ
E_3	$5.7e4$ KJ
ρ	1000 Kg/m^3
C_p	4.2 KJ/Kg K
V	0.01 m^3
F/V	0.1 min^{-1}
T_0	295 K
C_{A_0}	1 Kmol/min

we have,

$$\begin{aligned} \dot{\eta} = &-k_1(z_2)(\eta + C_{AS}) - k_3(z_2)(\eta + C_{As})^2 \\ &+ (C_{A0} - \eta - C_{AS})u_1, \end{aligned} \tag{64}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} k_1(z_2)(\eta + C_{AS}) - k_2(z_2)(z_1 + C_{BS}) \\ \dfrac{1}{\rho C_p}[\Delta H_1 k_1(z_2)(\eta + C_{AS}) \ldots \\ \ldots + \Delta H_2 k_2(z_2)(z_1 + C_{BS}) \\ +\Delta H_3 k_3(z_2)(\eta + C_{AS})^2] \end{bmatrix} + \begin{bmatrix} -(z_1 + C_{BS}) & 0 \\ \dfrac{T_0 - z_2 - T_S}{\rho C_p} & \dfrac{1}{V\rho C_p} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{65}$$

$$y = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{66}$$

Applying an I/O linearizing feedback based on the nominal values of the parameters and using Lemmas 1 and 2 to characterize the uncertainty in A_2, the following uncertain linear subsystem results:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0.00130\theta & 5.2\theta \times 10^{-5} \\ 0.01550\theta & 6.2\theta \times 10^{-4} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} W_d d. \tag{67}$$

Here θ is the uncertain parameter, v_1 and v_2 are the new input variables, d is the vector of perturbations due to uncertainty characterized as external disturbances and W_d is the bound on the magnitude of these disturbances.

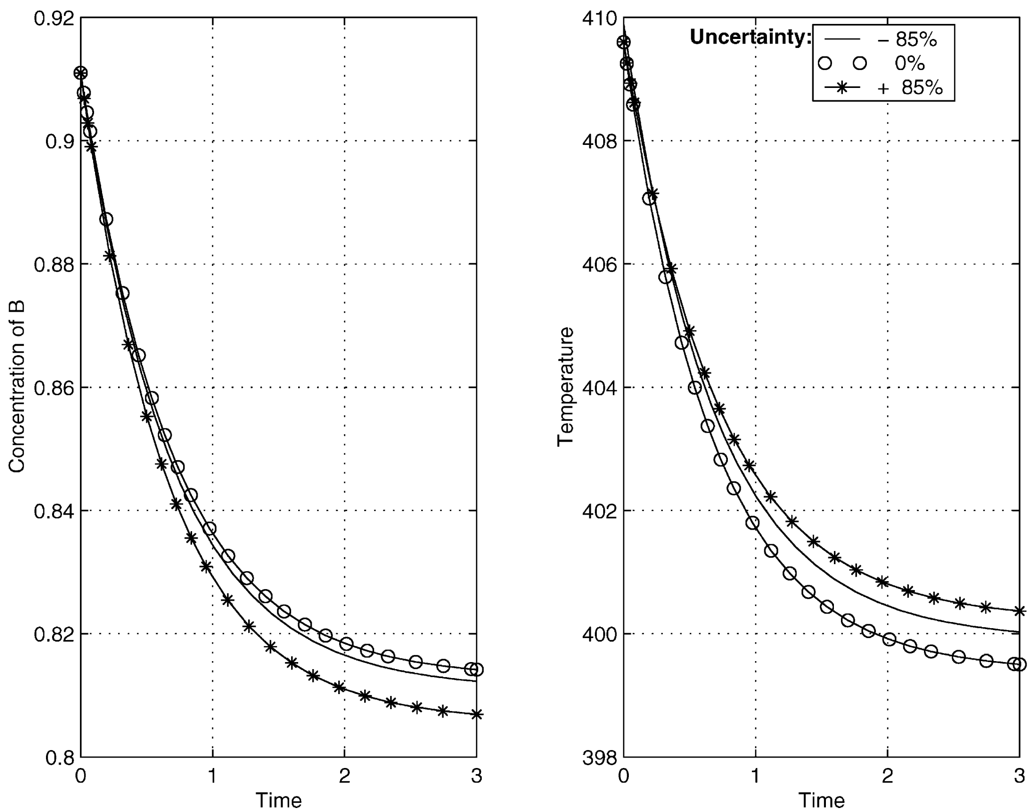

Fig. 2. Output profiles for conventional I/O controller.

If there is no parametric uncertainty (i.e. $d = 0$, and $\theta = \theta_0$), then a conventional I/O linearization strategy decouples the system into two linear SISO loops on which one can impose any desired linear dynamics. However, when the feedback law, designed assuming no uncertainty, is implemented on the perturbed system, exact linearization as well as decoupling is lost which could lead to performance degradation. Fig. 2 illustrates this loss of performance when the conventional I/O linearization design methodology is used. The controller is designed assuming that there is no uncertainty and the poles are placed at—1.5 and −1.5. The following feedback results, which is implemented on the plant.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} z_1 + C_{BS} & 0 \\ \dfrac{T_0 - z_2 - T_S}{\rho C_p} & \dfrac{1}{V\rho C_p} \end{bmatrix}^{-1} \times \begin{bmatrix} v_1 - k_1(\eta + C_{AS}) - k_2(z_1 + C_{BS}) \\ v_2 - \dfrac{1}{\rho C_p}(\Delta H_1 k_1(\eta + C_{AS}) \ldots \\ + \Delta H_2 k_2(z_1 + C_{BS}) + \Delta H_3 k_3(\eta + C_{AS})^2) \end{bmatrix}. \quad (68)$$

It is seen that when the plant parameters match the model parameters ($d = 0$), the set-points are tracked well. However, when the plant parameter, A_2, is perturbed by 85% the feedback law (68) does not track the desired outputs well.

Now, the robust controller described in the previous section is implemented on this example. The H_∞ objective is to minimize the influence of the disturbance vector on the output vector and the H_2 objective is to minimize the effect of the disturbance on the vector $[z_1 \ z_2 \ v_1 \ v_2]$ (LQG cost of control). Using the LMI control toolbox from MATLAB the following robust feedback was obtained.

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -11.0 & -0.001 \\ -0.001 & -11.0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \quad (69)$$

Simulations were carried out for the nonlinear system for different values of A_2 . It is observed that the robust I/O linearizing control is able to keep both the outputs at the desired setpoints despite the uncertainty in A_2. The results of the simulation are shown in Fig. 3. The efficacy of the robust controller to regulate the system starting from different initial conditions was also studied. The results are shown in Fig. 4. It is seen that the controller performs satisfactorily for a wide range of initial conditions.

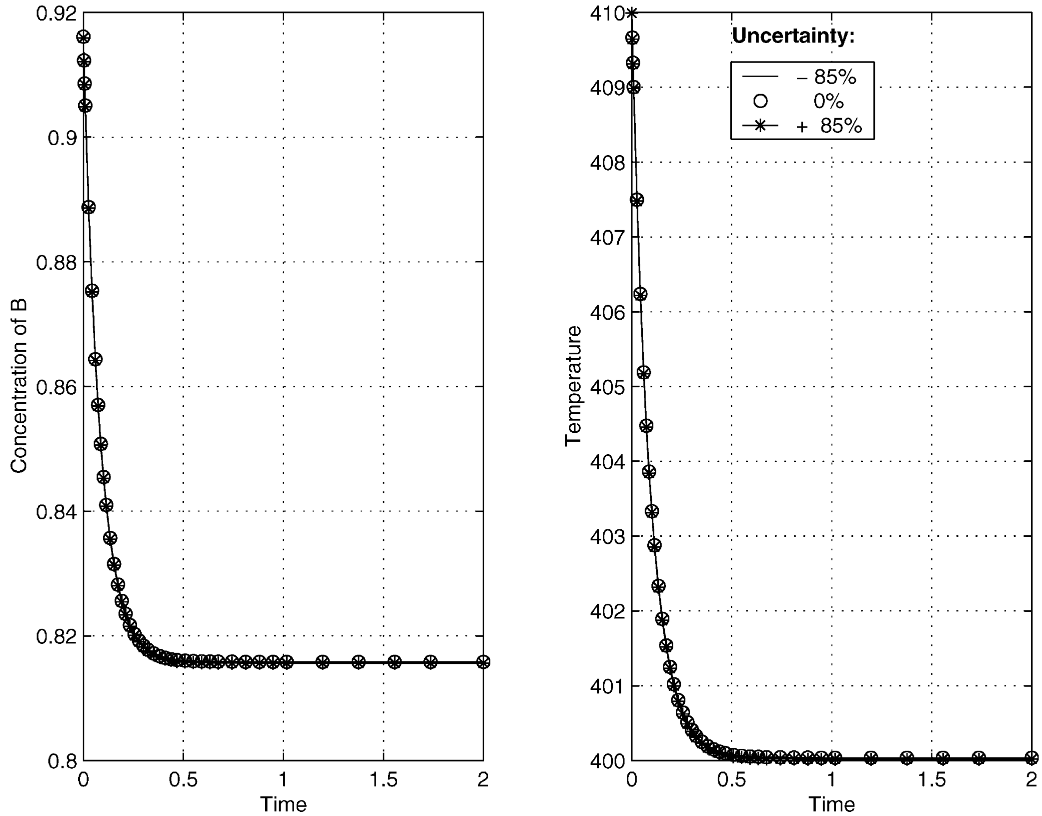

Fig. 3. Output profiles for robust controller ($x_0 = 0.15, 0.7, 390$).

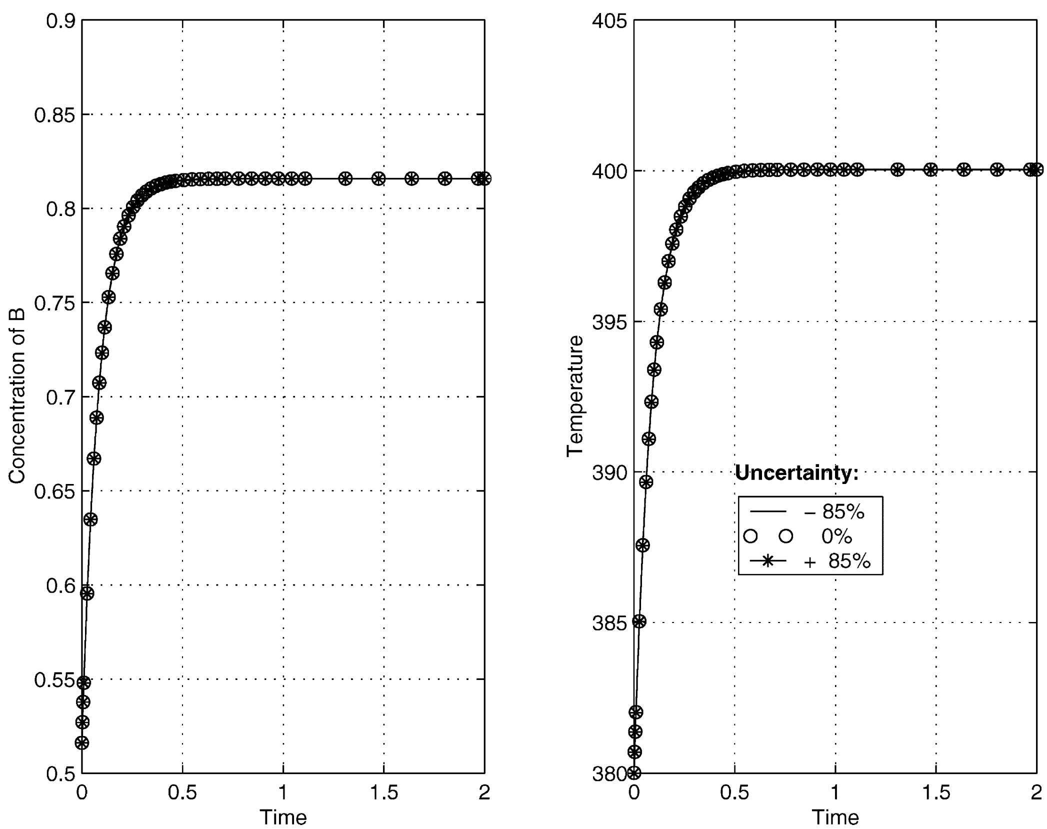

Fig. 4. Output profiles for robust controller ($x_0 = 0.15, 1.2, 412$).

5. Conclusions

A controller synthesis procedure was developed for minimum phase I/O linearizable, multivariable, nonlinear systems that are subject to parametric uncertainty. The controller that that results from this approach has a multi-loop structure. The inner loop implements a nonlinear transformation based on nominal parameters of the model. The outer loop is a multivariable controller that not only accounts for interacting channel effects but also provides robustness to uncertainties in plant parameters. The controllers can be designed using off-the-shelf software and do not require restrictive matching conditions to be satisfied. This methodology was illustrated via simulation of a regulation problem in a CSTR.

Notation

A_1, A_2	Arrhenius coefficients for rate constants
C_A, C_B	concentrations of the species A and B
C_p	specific heat
d	disturbance vector
E_1, E_2	activation energies
F	flowrate into the reactor
k_1, k_2	reaction rate constants
$L_f h$	Lie derivative of h along f
r_i	relative degree with respect to output y_i
T	temperature
$u_1 \ldots u_m$	original inputs
$v_1 \ldots v_m$	external reference inputs
V	volume of the reactor
W_d	weight on the disturbance
$x_1 \ldots x_n$	system states
z_i	states of the linearized subsystem
$y_1 \ldots y_m$	outputs

Greek letters

Δ	nonlinear perturbation
ΔH_1	enthalpy change of the first reaction
ΔH_2	enthalpy change of the second reaction
η_i	states of the zero dynamics
θ	set of uncertain parameters
ρ	density

Acknowledgements

Partial support of this research was provided by the Woodrow W. Everett, Jr. SCEEE Development Fund.

References

Chou, Y., & Wu, W. (1995). Robust controller design for uncertain nonlinear systems via feedback linearization. *Chemical Engineering Science*, *50*(9), 1429–1439.

Christofides, P., & Daoutidis, P. (1997). Robust control of multivariable two-time-scale nonlinear systems. *Journal of Processing and Control*, *7*, 313–328.

Christofides, P., Teel, A., & Daoutidis, P. (1996). Robust semi-global output tracking for singularly perturbed nonlinear systems. *International Journal of Control*, *65*, 639–666.

Daoutidis, P., & Kravaris, C. (1991). Inversion and zero dynamics in nonlinear multivariable control. *A.I.Ch.E. Journal*, *37*(4), 527–538.

Daoutidis, P., Soroush, M., & Kravaris, C. (1990). Feedforward/feedback control of multivariable nonlinear processes. *A.I.Ch.E. Journal*, *36*, 1471–1484.

Enns, E., Bugajski, D., Hendrick, R., & Stein, G. (1994). Dynamic inversion: An evolving methodology for flight control design. *International Journal of Control*, *59*(1), 71–79.

Gahinet, P., Nemirovski, A., Laub, A., & Chilali, M. (1995). *LMI Control Toolbox*, Natick, MA, The Mathworks Inc.

Isidori, A. (1995). *Nonlinear control systems*. Berlin: Springer.

Isidori, A., & Ruberti, A. (1984). On the synthesis of linear input–output responses for nonlinear systems. *Systems and Control Letters*, *4*, 17–22.

Khargonekar, P., & Rotea, M. (1991). Mixed h_2/h_∞ control: a convex optimation approach. *IEEE Transactions on Automation Control*, *AC-39*, 824–837.

Kolavennu, S., Palanki, S., & Cockburn, J. (2000). Robust control of *i/o* linearizable systems via multi-model h_2/h_∞ synthesis. *Chemical Engineering Science*, *55*(9), 1583.

Kravaris, C., & Palanki, S. (1988). Robust nonlinear state feedback under structured uncertainty. *A.I.Ch.E. Journal*, *34*(7), 1119–1127.

Morton, B. (1995). Dynamic inversion control approach for high-mach trajectory tracking. *Proceedings of the American Control Conference*, (vol. 2), pp. 1332–1336.

Nijmeijer, H., & der Schaft, A. V. (1990). *Nonlinear dynamical control systems.*. Berlin: Springer.

Palanki, S., & Kravaris, C. (1997). Controller synthesis for time-varying systems by input/output linearization. *Computers and Chemical Engineering*, *21*(8), 891–903.

Sampath, V., Palanki, S., & Cockburn, J. C. (1998). Robust nonlinear control of polymethylmethacrylate production in a batch reactor. *Computers and Chemical Engineering*, *22*, S451–S457.

Sampath, V., Palanki, S., & Cockburn, J. C. (1999). Development of robust gain scheduling for trajectory tracking problems in batch processes. *Computers and Chemical Engineering*, *23*, 277–282.

Sastry, S., & Isidori, A. (1989). Adaptive control of linearizable systems. *IEEE Transactions on Automation Control*, *34*, 1123–1131.

Singh, S. N., & Rugh, W. J. (1972). Decoupling in a class of nonlinear systems by state variable feedback. *Transactions of ASME Journal of Dynamic Systems Measures and Control*, *94*, 323–329.

Soroush, M., & Kravaris, C. (1994). Nonlinear control of a polymerization cstr with singular characteristic matrix. *A.I.Ch.E. Journal*, *40*(6), 980–990.