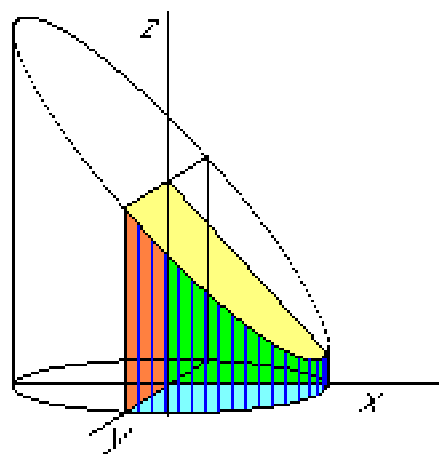
Page 549, #21(c)

1 p549, #21(c), §1 Asked

Asked: Find the centroid of the first octant region inside $x^2 + y^2 = 9$ and below $x + z = 4$.

2 p549, #21(c), §2 Approach

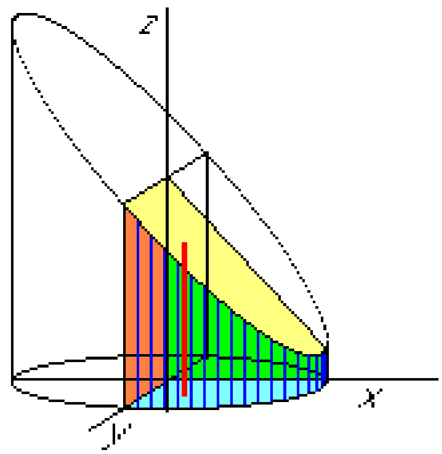
The region inside $x^2 + y^2 = 9$ is the inside of a cylinder of radius 3 around the z-axis. The equation $x + z = 4$ describes a plane through the y-axis under 45 degrees with the x-axis:



Use cylindrical coordinates r, θ, and z:

$$x = r\cos\theta \qquad y = r\sin\theta$$

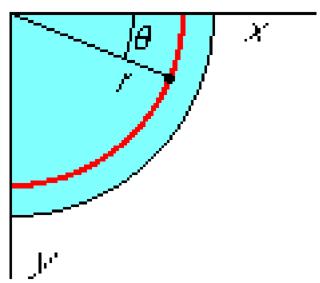
Integrate z first:



(Why not r first? Why not θ?). Boundaries are

$$z_1 = 0 \qquad z_2 = 4 - x = 4 - r\cos\theta$$

Next integrate θ and r:



$$\theta_1 = 0 \qquad \theta_2 = \tfrac{1}{2}\pi$$

$$r_1 = 0 \qquad r_2 = 3$$

3 p549, #21(c), §3 Results

For the volume $V = \int\int\int \, \mathrm{d}V = \int\int\int r \; \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$:

$$V = \int_{\theta=0}^{\pi/2} \int_{r=0}^{3} \left[\int_{z=0}^{4-r\cos\theta} r \; \mathrm{d}z \right] \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{3} (4 - r\cos\theta) r \, \mathrm{d}r \right] \, \mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2} 18 - 9\cos\theta \, \mathrm{d}\theta = 9(\pi - 1)$$

For $V\bar{x} = \int\int\int x \ \mathrm{d}V = \int\int\int xr \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$:

$$V\bar{x} = \int_{\theta=0}^{\pi/2} \int_{r=0}^{3} \left[\int_{z=0}^{4-r\cos\theta} r^2 \cos\theta \ \mathrm{d}z \right] \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{3} 4r^2\cos\theta - r^3\cos^2\theta \, \mathrm{d}r \right] \, \mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2} 36\cos\theta - \frac{81}{4}\cos^2\theta \, \mathrm{d}\theta = \frac{9}{16}(64 - 9\pi)$$

hence $\bar{x} = (64 - 9\pi)/16(\pi - 1)$

Etcetera.