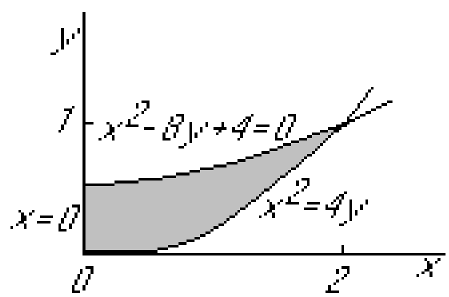
Page 528, #14(e)

1 p528, #14(e), §1 Asked

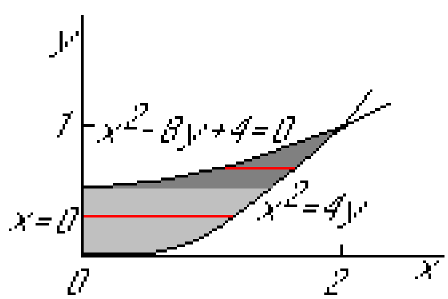
Asked: Find the centroid of the first-quadrant area bounded by $x^2 - 8y + 4 = 0$ and $x^2 = 4y$ and $x = 0$. (Slighty different from the book.)

2 p528, #14(e), §2 Region



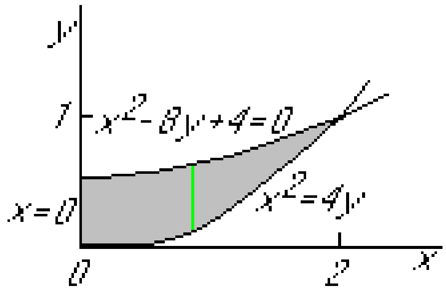
3 p528, #14(e), §3 Approach

Integrate x first?



The integral would have to be split up into the light and dark areas since the lower boundary of integration is $x = 0$ in the light region and $x = \sqrt{8y - 4}$ in the dark region.

Integrate y first!



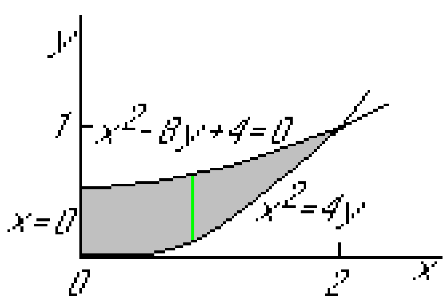
The boundaries of integration will be

$$y_1 = \tfrac{1}{4}x^2 \qquad y_2 = \tfrac{1}{8}x^2 + \tfrac{1}{2}$$

After integration over y, the remaining region of integration over x will be a line segment:

$$x_1 = 0 \qquad x_2 = 2$$

4 p528, #14(e), §4 Results



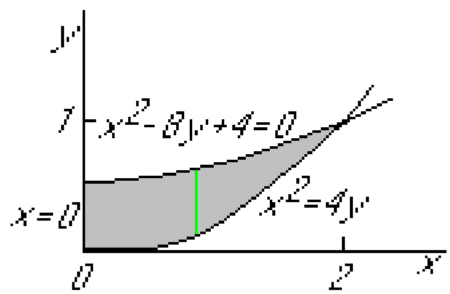
For $A = \int \,\mathrm{d}A = \int\int \,\mathrm{d}x\,\mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \,\mathrm{d}y \right] \,\mathrm{d}x$$

$$= \int_{x=0}^{2} \left[y\Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \, \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8}x^2 + \tfrac{1}{2}) - (\tfrac{1}{4}x^2) \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{2} - \tfrac{1}{8}x^2 \right] \mathrm{d}x = \tfrac{2}{3}$$



For $A\bar{x} = \int x \, \mathrm{d}A = \int \int x \, \mathrm{d}x \, \mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} x \ \mathrm{d}y \right] \, \mathrm{d}x$$

where x is constant in the integration;

$$= \int_{x=0}^{2} \left[xy\Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \, \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8}x^3 + \tfrac{1}{2}x) - (\tfrac{1}{4}x^3) \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{2}x - \tfrac{1}{8}x^3 \right] \mathrm{d}x = \tfrac{1}{2}$$

Hence $\bar{x} = \frac{1}{2}/\frac{2}{3} = \frac{3}{4}$.

For $A\bar{y} = \int y \, \mathrm{d}A = \int \int y \, \mathrm{d}x \, \mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} y \ \mathrm{d}y \right] \ \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[\tfrac{1}{2} y^2 \Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \ \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[\tfrac{1}{2} (\tfrac{1}{8}x^2 + \tfrac{1}{2})^2 - \tfrac{1}{2} (\tfrac{1}{4}x^2)^2 \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8} + \tfrac{1}{16}x^2 - \tfrac{3}{128}x^2 \right] \mathrm{d}x = \tfrac{4}{15}$$

Hence $\bar{y} = \frac{4}{15} / \frac{2}{3} = \frac{2}{5}$.