Introduction

Optimization:

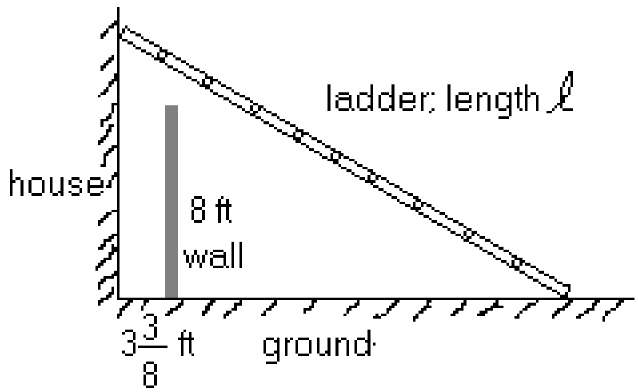
- best design;
- drag reduction;
- potential energy minimization;
- economics;
- ...

Key ideas:

- zero partial derivatives at an interior extremum
- Lagrangian multipliers can account for constraints

Page 127, #30

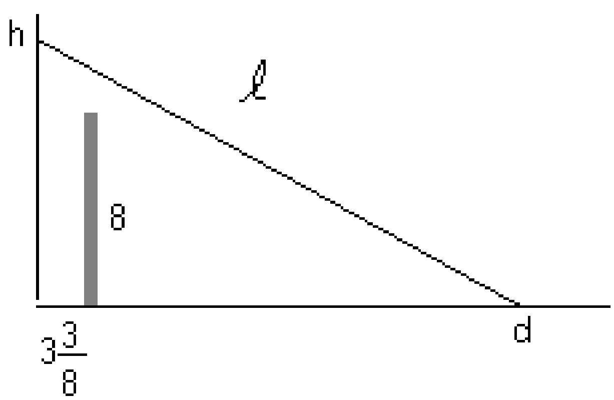
1 p127, #30, §1 Asked



Given: A free standing wall, located $3\frac{3}{8}$ ft from the side of a house.

Asked: What is the length ℓ of the shortest ladder that can reach the house (over the free standing wall).

2 p127, #30, §2 Definition

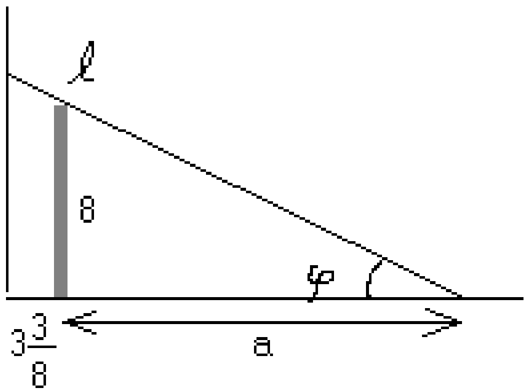


Two degrees of freedom: say h and d

One *inequality* constraint: the ladder must be above the free standing wall.

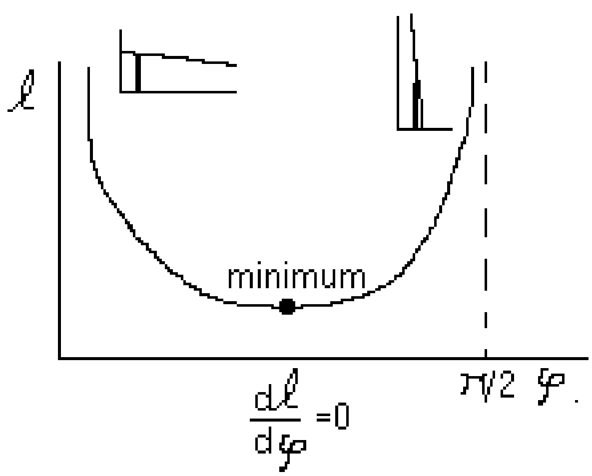
3 p127, #30, §3 Reduction

The shortest ladder hits the free standing wall:



One degree of freedom left: φ.

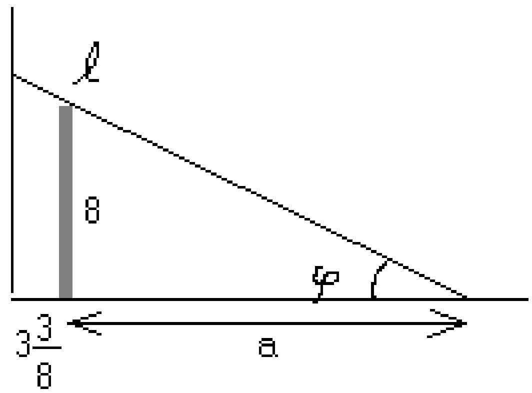
4 p127, #30, §4 Further Reduction



At the minimum:

$$\frac{d\ell}{d\varphi} = 0 \tag{1}$$

5 p127, #30, §5 Finding l



First find a:

$$a = \frac{8}{\tan\varphi}. \tag{2}$$

Then:

$$\ell = \frac{3\frac{3}{8} + a}{\cos\varphi} = \frac{3\frac{3}{8}}{\cos\varphi} + \frac{8}{\sin\varphi} \tag{3}$$

6 p127, #30, §6 Solving l'=0

$$\frac{\mathrm{d}\ell}{\mathrm{d}\varphi} = \frac{3\frac{3}{8}}{\cos^2\varphi}\sin\varphi - \frac{8}{\sin^2\varphi}\cos\varphi = 0. \tag{4}$$

$$\frac{27}{8\cos^2\varphi}\sin\varphi = \frac{8}{\sin^2\varphi}\cos\varphi \tag{5}$$

$$\tan^3\varphi = \frac{64}{27} \qquad \Longrightarrow \qquad \varphi_{\text{min}} = 0.9273 \text{ radians} \tag{6}$$

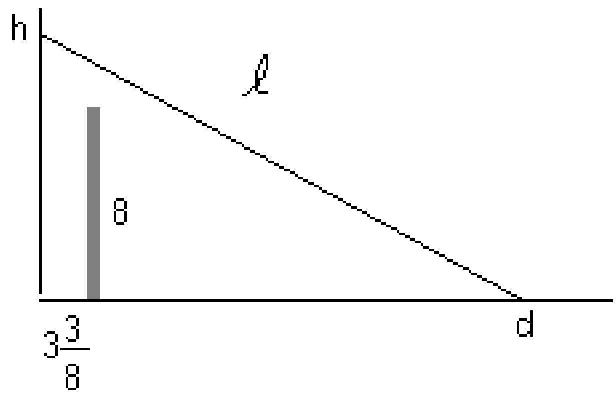
7 p127, #30, §7 Finding l

From (3)

$$\ell_{\text{min}} = 15.625 \text{ ft} \tag{7}$$

#30, General Method

1 p127, #30[alt], §1 Definition

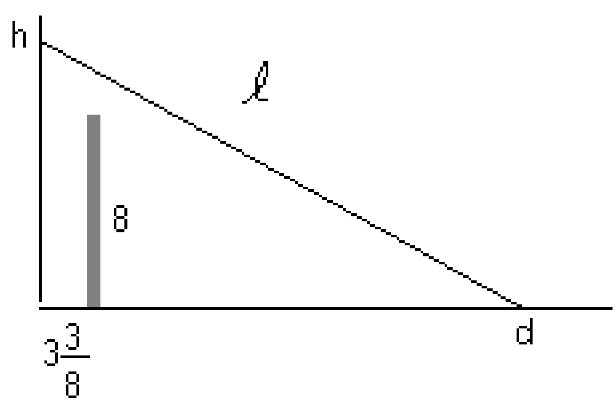


Two degrees of freedom: h and d

One *inequality* constraint (from similar triangles):

$$h\frac{d - 3\frac{3}{8}}{d} > 8 \qquad \Longrightarrow \qquad h[d - 3\frac{3}{8}] - 8d > 0 \tag{1}$$

2 p127, #30[alt], §2 Formulation



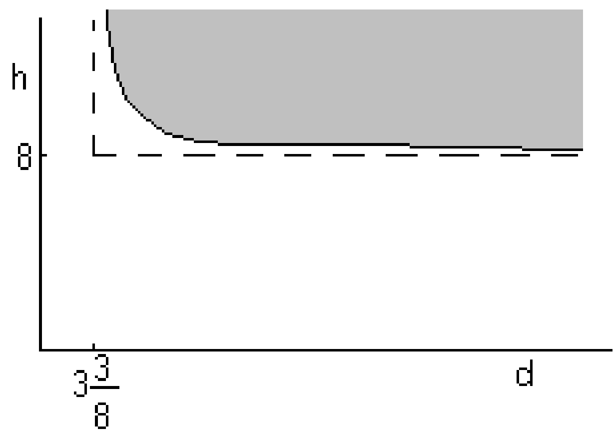
Minimize

$$\ell(h, d) = \sqrt{h^2 + d^2} \tag{2}$$

(from Pythagoras), subject to

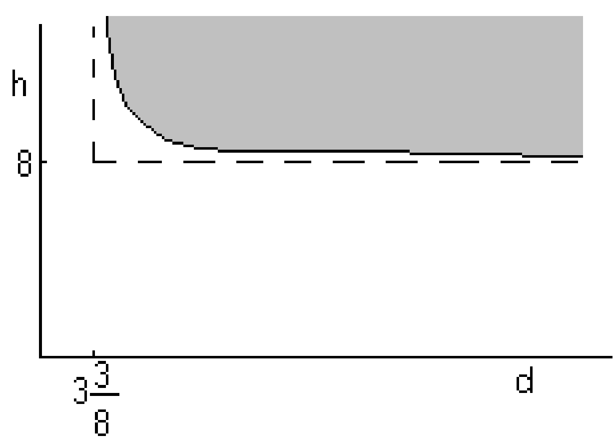
$$h[d - 3\tfrac{3}{8}] - 8d > 0 \tag{3}$$

3 p127, #30[alt], §3 Interior Minima



$$\frac{\partial \ell}{\partial d} = 0 \qquad \frac{\partial \ell}{\partial h} = 0 \qquad \Longrightarrow \qquad d = h = \ell = 0 \tag{4}$$

4 p127, #30[alt], §4 Boundary Minima



Use a Lagrangian multiplier for the constraint

$$f = \sqrt{h^2 + d^2} + \lambda(h[d - 3\tfrac{3}{8}] - 8d). \tag{5}$$

Search for an unconstrained stationary point:

$$\frac{\partial f}{\partial d} = 0 \qquad \frac{\partial f}{\partial h} = 0 \qquad \frac{\partial f}{\partial \lambda} = 0 \tag{6}$$

Introduction

Graphs:

- Understanding,
- Summarizing data,
- Representing data,
- Interpolating data,
- ...

Need maxima and minima ($y' = 0$, relative or absolute), inflection points ($y'' = 0$), vertical asymptotes ($y \to \infty, x$ finite), horizontal asymptotes ($x \to \infty, y$ finite), oblique asymptotes ($y \propto x \to \infty$), behavior at infinity ($x \to \infty$), intercepts (x or $y = 0$), singular points (corners, cusps, crossings, infinite curvature, ...), concavity (upward if $y'' > 0$), symmetry or anti-symmetry around x, y, or general oblique axes, ...) See page 133 in the 4th edition of Ayres.

If you can, draw the curve first, then fill in the details.

Page 139, #13(a)

1 p139, #13(a), §1 Asked

Asked: Draw the graph of

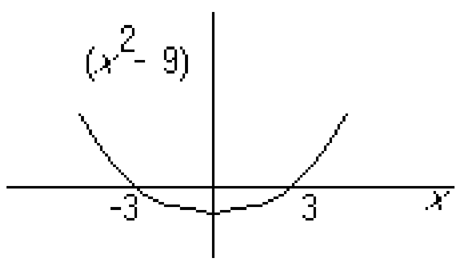
$$xy = \left(x^2 - 9\right)^2 \tag{1}$$

2 p139, #13(a), §2 Graph

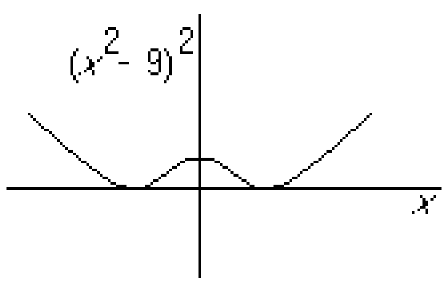
$$xy = \left(x^2 - 9\right)^2 \tag{2}$$

Instead of starting to crunch numbers, look at the pieces first:

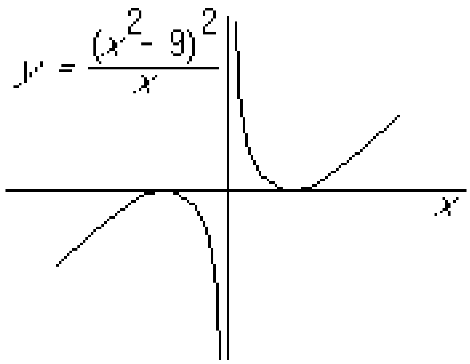
Factor $x^2 - 9 = (x-3)(x+3)$ is a parabola with zeros at $x = \pm 3$:



Squaring gives a quartic with double zeros at $x = \pm 3$:



Dividing by x will produce a simple pole at $x = 0$ and also a sign change at negative x:



Function $y(x)$:

- has an x-extent $x \neq 0$ and a y-extend $-\infty < y < \infty$;
- is odd (symmetric with respect to the origin);
- has a relative maximum at -3 of finite curvature: $y \propto (x+3)^2$;
- has a relative minimum at 3 of finite curvature: $y \propto (x-3)^2$;
- has a vertical asymptote at $x = 0$ with asymptotic behavior: $y \sim 81/x$ for $|x| \to 0$;
- behaves asymptotically as $y \sim x^3$ for $|x| \to \infty$;
- is concave up for $x > 0$, down for $x < 0$

3 p139, #13(a), §3 Alternate

$$y = \frac{(x^2 - 9)^2}{x}$$

Hence

- intercepts with x-axis are at $x = \pm 3$;
- no intercepts with the y axis;
- y is an odd function of x (symmetric about the origin);
- for $x \downarrow 0$, $y \to \infty$ (vertical asymptote);

- for $x \uparrow 0$, $y \to -\infty$ (singularity is an odd, simple pole);
- for $x \to \pm\infty$, $y \sim x^3 \to \pm\infty$.

$$y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2 - 9)(3x^2 + 9)}{x^2}$$

Hence,

- $y' > 0$ for $-\infty < x < -3$ (y increases from $-\infty$);
- $y' = 0$ for $x = -3$ (local maximum, $y = 0$);
- $y' < 0$ for $-3 < x < 0$ (y decreases towards $-\infty$);
- $y' = -\infty$ for $x = 0$ (singular point, vertical asymptote);
- $y' < 0$ for $0 < x < -3$ (decreases from ∞);
- $y' = 0$ for $x = 3$ (local minimum, $y = 0$);
- $y' > 0$ for $3 < x < \infty$ (increases to ∞).

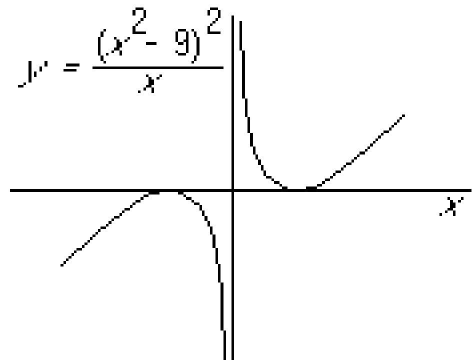
Also,

- $y' \to \infty$ when $x \to \pm\infty$ (no horizontal or oblique asymptotes);
- all derivatives exist, except at $x = 0$, which has no point on the curve (no corners, cusps, infinite curvature, or other singular points);
- probably no inflection points.

$$y'' = \frac{6x^4 + 162}{x^3}$$

Hence

- really no inflection points (since there is no point at $x = 0$);
- cocave downward for $x < 0$, upward for $x > 0$.



Hence the x- and y-extends as before.

Page 139, #13g

1 p139, #13g, §1 Asked

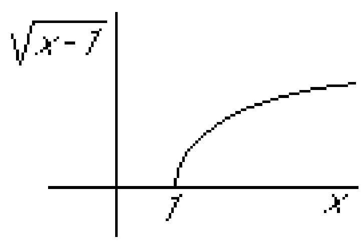
Asked: Graph

$$y = x\sqrt{x-1} \tag{1}$$

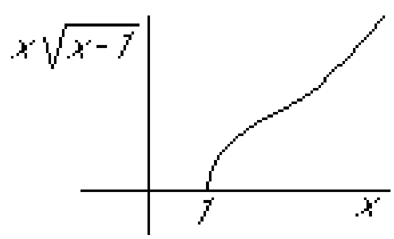
2 p139, #13g, §2 Solution

$$y = x\sqrt{x-1} \tag{2}$$

Factor $\sqrt{x-1}$ is $\sqrt{x}$ shifted one unit towards the right.



Multiplying by x magnifies it by a factor ranging from 1 to ∞:



Function $y(x)$:

- has an x-extent $x \geq 1$ and a y-extent $y \geq 0$;
- behaves asymptotically as $y \sim x^{3/2}$ for $x \to \infty$;
- is monotonous:

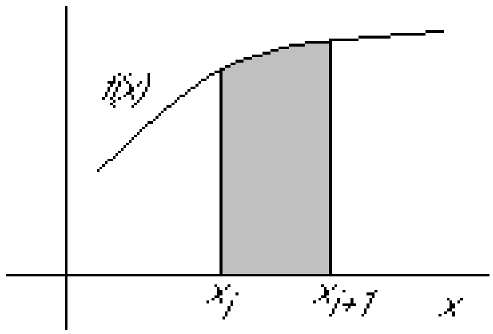
$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x-1} + \frac{x}{2\sqrt{x-1}} = \frac{2x-2+x}{2\sqrt{x-1}} = \frac{3x-2}{2\sqrt{x-1}} > 0;$$

- has vertical slope at $x = 1$;
- is concave down for smaller x, concave up for larger x;
- the inflection point is at

$$y'' = \frac{3x - 4}{4(x-1)^{3/2}} = 0$$

giving $x = 4/3$.

Introduction

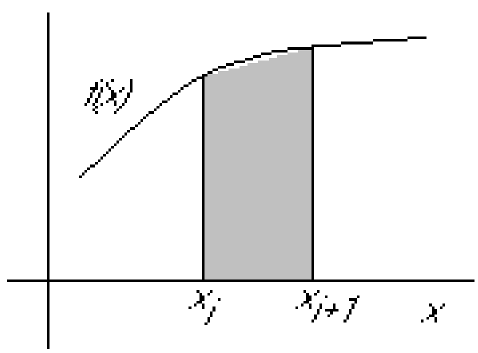


Numerical integration using Newton formulae:

- can handle any function;
- simple;
- can handle measured data easily.

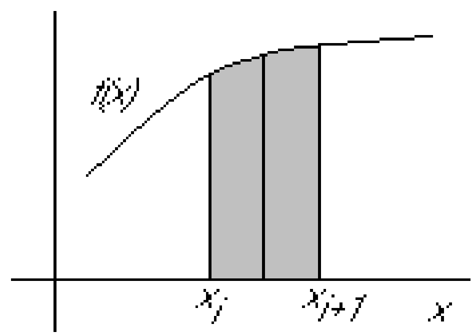
Trapezium rule for an interval from $x = x_i$ to x_{i+1}:

$$\int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x \approx (x_{i+1} - x_i)\,\frac{f(x_i) + f(x_{i+1})}{2}$$



Simpson rule for an interval from $x = x_i$ to x_{i+1}:

$$\int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x \approx (x_{i+1} - x_i)\,\frac{f(x_i) + 4f(x_{i+\frac{1}{2}}) + f(x_{i+1})}{6}$$



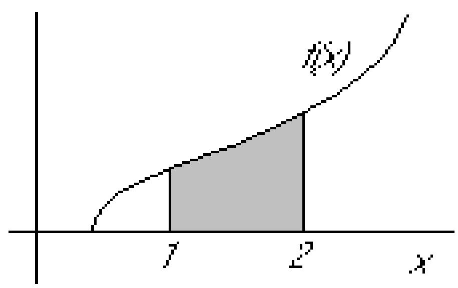
These rules are accurate if the interval from x_i to x_{i+1} is sufficiently small. To integrate over an interval that is not small, divide it into small ones, then integrate over each small interval and add the results.

Page 224, #44 (mod)

1 p224, #44(mod), §1 Asked

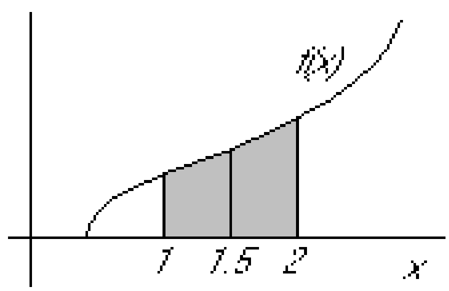
Asked:

$$\int_1^2 x\sqrt[3]{x^5+2x^2-1}\ \mathrm{d}x$$



2 p224, #44 (mod), §2 Solution

Divide into n=2 intervals and use the trapezium rule:



If

$$f(x) = x\sqrt[3]{x^5+2x^2-1}$$

then the trapezium rule gives

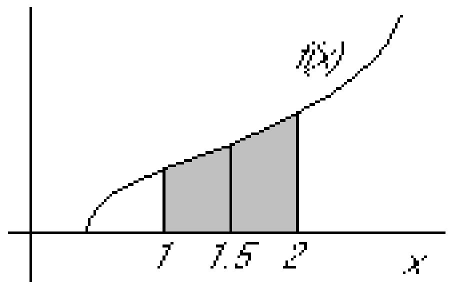
$$\int_1^{1.5} f\ \mathrm{d}x = 0.5\frac{f(1)+f(1.5)}{2} = 0.5\frac{1.259921+3.345421}{2} = 1.151336$$

$$\int_{1.5}^{2} f\ \mathrm{d}x = 0.5\frac{f(1.5)+f(2)}{2} = 0.5\frac{3.345421+6.782423}{2} = 2.531961$$

$$\int_1^2 f \; \mathrm{d}x = 1.151336 + 2.531961 = 3.683297$$

Exact is 3.571639.

Now divide into n=2 *half* intervals and use the Simpson rule:



$$\int_1^2 f \; \mathrm{d}x = 1\frac{f(1) + 4f(1.5) + f(2)}{6}$$

$$= 1\frac{1.259921 + 4 * 3.345421 + 6.782423}{6} = 3.570671$$

Closer to the exact value 3.571639.

Introduction

Limits:

- approximation;
- order estimates;
- function evaluation;
- stagnation streamlines;
- ...

Page 250, #10(v)

1 p250, #10(v), §1 Asked

Asked:

$$\lim_{x \to -\infty} x^2 e^x \tag{1}$$

2 p250, #10(v), §2 Observations

$$\lim_{x \to -\infty} x^2 e^x$$

$$x^2 \to \infty \qquad e^x \to 0$$

3 p250, #10(v), §3 L'Hopital

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}} = \lim_{x \to -\infty} \frac{(x^2)'}{(e^{-x})'} = \lim_{x \to -\infty} \frac{2x}{-e^{-x}} = \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0$$

4 p250, #10(v), §4 Better

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}} = 0$$

since $e^{|x|}$ is greater than any power of x for large $|x|$.

Page 250, #10(z)

1 p250, #10(z), §1 Asked

Asked:

$$\lim_{x\to 0}(x - \arcsin x)\csc^3 x \tag{1}$$

2 p250, #10(z), §2 Procedure

L'Hopital:

$$\lim_{x\to 0}(x - \arcsin x)\csc^3 x = \lim_{x\to 0}\frac{x - \arcsin x}{\sin^3 x} \tag{2}$$

L'Hopital:

$$\lim_{x\to 0}\frac{(x - \arcsin x)'''}{(\sin^3 x)'''} = ? \tag{3}$$

3 p250, #10(z), §3 Simpler

Since $\sin x \approx x$ for small x, $\sin^3 x \approx x^3$. Also $\arcsin x \approx x + \frac{1}{6}x^3$. So

$$\lim_{x\to 0}\frac{x - \arcsin x}{\sin^3 x} \approx \frac{-\frac{1}{6}x^3}{x^3} = -\frac{1}{6}$$

Introduction

Curvilinear motion:

- Dynamics of vehicles (cars, planes, ...)
- Ballistics,
- Forces,
- Vortex lines,
- ...

$$\vec{r} = \vec{r}(t) \qquad \vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \qquad \vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

Page 369, #14

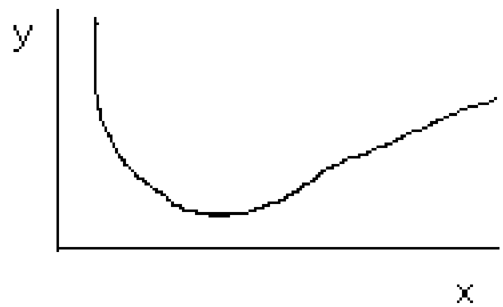
1 p369, #14, §1 Asked

Given: A particle moves along a curve described by

$$x = \tfrac{1}{2}t^2 \qquad y = \tfrac{1}{2}x^2 - \tfrac{1}{4}\ln x \tag{1}$$

Asked: The velocity and acceleration at $t = 1$

2 p369, #14, §2 Graphically



3 p369, #14, §3 Position

At $t = 1$:

$$x = \tfrac{1}{2}t^2 = \tfrac{1}{2} \qquad y = \tfrac{1}{2}x^2 - \tfrac{1}{4}\ln x = 0.298 \tag{2}$$

hence

$$\vec{r} = \begin{pmatrix} 0.5 \\ 0.298 \end{pmatrix} = 0.5\hat{\imath} + 0.298\hat{\jmath} \tag{3}$$

4 p369, #14, §4 Velocity

Velocity:

$$\vec{v} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}x}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} t \\ (x - \frac{1}{4}x^{-1})t \end{pmatrix} = \begin{pmatrix} t \\ \frac{1}{2}t^3 - \frac{1}{2}t^{-1} \end{pmatrix} \tag{4}$$

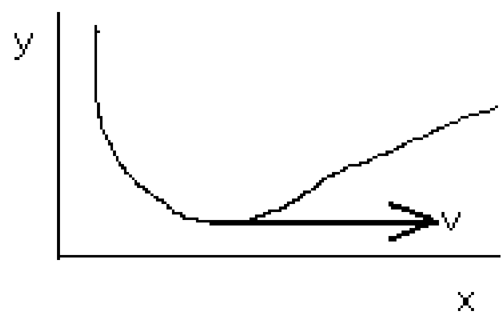
Velocity at $t = 1$:

$$\vec{v}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\hat{\imath} + 0\hat{\jmath} = \hat{\imath} \tag{5}$$

Components at $t = 1$:

$$v_x \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = 1 \qquad v_y \equiv \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \tag{6}$$

5 p369, #14, §5 Graphically



6 p369, #14, §6 Properties

Magnitude at $t = 1$:

$$|\vec{v}| = v \equiv \frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{v_x^2 + v_y^2} = 1 \tag{7}$$

Angle with the positive x-axis at $t = 1$:

$$\tau = \arctan \frac{v_y}{v_x} = 0 \text{ (not } \pi\text{).} \tag{8}$$

7 p369, #14, §7 Acceleration

Acceleration:

$$\vec{a} = \begin{pmatrix} \dfrac{\mathrm{d}v_x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}v_y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2}t^2 + \frac{1}{2}t^{-2} \end{pmatrix} \tag{9}$$

from (4).

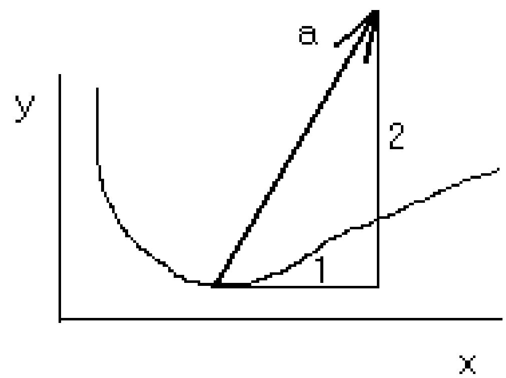
Acceleration at $t = 1$:

$$\vec{a}(1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\hat{\imath} + 2\hat{\jmath} \tag{10}$$

Components at $t = 1$:

$$a_x \equiv \frac{\mathrm{d}v_x}{\mathrm{d}t} = 1 \qquad a_y \equiv \frac{\mathrm{d}v_y}{\mathrm{d}t} = 2 \tag{11}$$

8 p369, #14, §8 Graphically



9 p369, #14, §9 Properties

Magnitude at $t = 1$:

$$|\vec{a}| = a = \sqrt{a_x^2 + a_y^2} = \sqrt{5} \tag{12}$$

Angle with the positive x-axis at $t = 1$:

$$\phi = \arctan \frac{a_y}{a_x} = 63° \text{ (not } 243°\text{).} \tag{13}$$

Component tangential to the motion:

$$a_t \equiv \frac{\mathrm{d}v}{\mathrm{d}t} \equiv \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|} = \frac{a_x v_x + a_y v_y}{|\vec{v}|} = 1 \tag{14}$$

Component normal to the motion:

$$a_n \equiv \frac{v^2}{R} = \sqrt{a^2 - a_t^2} = 2 \tag{15}$$

Introduction

Approximation:

- effort;
- accuracy;
- insight;
- ...

Page 438, #10(b)

1 p438, #10(b), §1 Asked

Asked: The Maclaurin series of $\sin^2 x$.

2 p438, #10(b), §2 Identification

General Taylor series:

$$\begin{aligned} f(x) &= f(a) + f'(a)\frac{x-a}{1!} + f''(a)\frac{(x-a)^2}{2!} + \ldots \\ &= \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!} \end{aligned}$$

This is a power series (a is a given constant.) Maclaurin series: $a = 0$.

Approach:

- note that $a = 0$;
- identify the derivatives;
- evaluate them at $a = 0$;
- put in the formula;
- identify the terms for any value of n.

3 p438, #10(b), §3 Results

$$\begin{array}{ll} f(x) = \sin^2 x & f(0) = 0 \\ f'(x) = 2\sin x \cos x & f'(0) = 0 \\ f''(x) = 2\cos^2 x - 2\sin^2 x = 2 - 4\sin^2 x & f''(0) = 2 \\ f'''(x) = -8\sin x \cos x = -4f'(x) & f'''(0) = 0 \\ f''''(x) = -4f''(x) & f''''(0) = -8 \\ f^{(5)}(x) = -4f'''(x) & f^{(5)}(0) = 0 \\ f^{(6)}(x) = -4f''''(x) = (-4)^2 f''(x) & f^{(6)}(0) = 32 \\ \vdots & \vdots \end{array}$$

$$\begin{aligned}\sin^2 x &= f(0) + f'(0)\frac{x-a}{1!} + f''(0)\frac{(x-a)^2}{2!} + \dots \\ &= 2\frac{x^2}{2!} - 8\frac{x^4}{4!} + 32\frac{x^6}{6!} + \dots\end{aligned}$$

General expression:

$$\text{When } n = 2k \text{ with } k \geq 1\text{: } f^{(n)} = 2(-4)^{k-1} \text{ Otherwise: } f^{(n)} = 0$$

$$\sin^2 x = \sum_{k=1}^{\infty} 2(-4)^{k-1}\frac{x^{2k}}{(2k)!}$$

4 p438, #10(b), §4 Other way

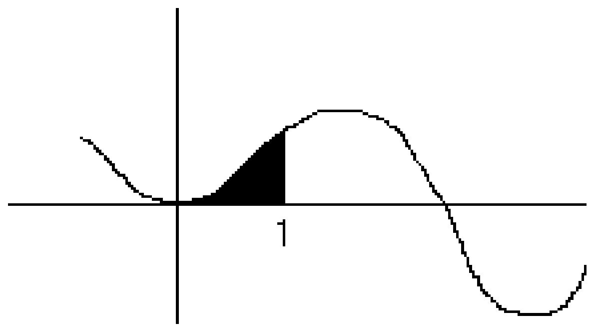
Write $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ and look up the Maclaurin series for the cosine. (No fair.)

Page 440, #30

1 p440, #30, §1 Asked

Asked: The area below $y = \sin x^2$ for $0 \leq x \leq 1$.

2 p440, #30, §2 Identification



$$\int_0^1 \sin x^2 \,\mathrm{d}x$$

Analytically?

Approximate $\sin x^2$ using a Taylor series.

3 p440, #30, §3 Finish

$$\begin{aligned} \textstyle\int_0^1 \sin x^2 \,\mathrm{d}x &= \int_0^1 \frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \ldots \\ &= \frac{1}{3} - \frac{1}{3!7} + \frac{1}{5!11} \\ &= .3103 \pm 0.0008 \end{aligned}$$

Introduction

Total differentials:

- error estimates;
- changes in compound quantities;
- ...

For $f = f(x, y, z)$,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$$

Page 461, #27(a)

1 p461, #27(a), §1 Asked

Given:

$$\omega = \sqrt[3]{\frac{g}{b}}$$

The maximum error in g is 1%, the maximum error in b is 0.5%.

Asked: The maximum percentage error in ω.

2 p461, #27(a), §2 Identification

Given are the relative errors:

$$\frac{\delta g}{g} = 0.01 \qquad \frac{\delta b}{b} = 0.005$$

Error manipulation rules:

1. During addition and substraction, add absolute errors;
2. During multiplication and division, add relative errors;
3. During exponentiation, multiply the relative error by the power.

3 p461, #27(a), §3 Results

$$\frac{\mathrm{d}g/b}{g/b} = \frac{b}{g}\left(\frac{b\mathrm{d}g - g\mathrm{d}b}{b^2}\right) = \frac{\mathrm{d}g}{g} - \frac{\mathrm{d}b}{b}$$

Hence the greatest possible relative error in g/b is:

$$\frac{\delta g/b}{g/b} = 0.01 + 0.005$$

(or use rule 2)

$$\frac{\mathrm{d}\sqrt[3]{g/b}}{\sqrt[3]{g/b}} = \tfrac{1}{3}\frac{\mathrm{d}g/b}{g/b}$$

(or use rule 3)

Hence

$$\frac{\delta\omega}{\omega} = 0.005 = 0.5\%$$

Page 461, #29

1 p461, #29, §1 Asked

Given: A circular cylinder of varying radius r and height h. At a given time, $r = 6$ inch, $\dot{r} = 0.2$ in/sec, $h = 8$ in, $\dot{h} = -0.4$ in/sec.

Asked: $\dot{V}$ and $\dot{A}$ at that time.

2 p461, #29, §2 Solution

$$V = \pi r^2 h \qquad A = 2\pi r h + 2\pi r^2$$

$$\mathrm{d}V = \frac{\partial V}{\partial h}\,\mathrm{d}h + \frac{\partial V}{\partial r}\,\mathrm{d}r$$

$$\dot{V} = \pi r^2 \dot{h} + \pi 2 r h \dot{r} = 15.08 \text{ in}^3/\text{sec}$$

$$\dot{A} = 2\pi r \dot{h} + (2\pi h + 4\pi r)\,\dot{r} = 10.05 \text{ in}^2/\text{sec}$$

Introduction

Vectors for geometry:

- straight line trajectories;
- surfaces;
- ...

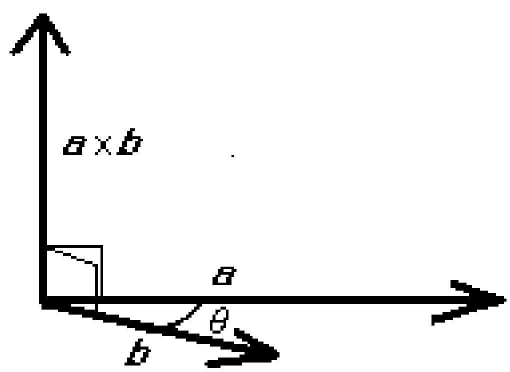
- Dot (scalar) product:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = |\vec{a}||\vec{b}| \cos \vartheta$$

- Cross (vector) product:

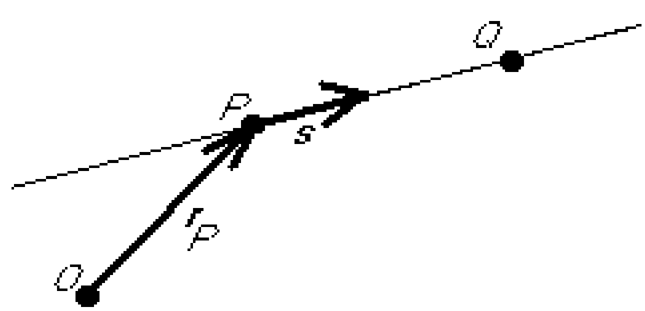
$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \vartheta$$

and normal to both vectors. Seen from below:



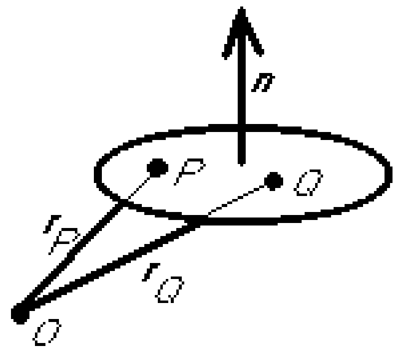
- Line through point P parallel to vector $\vec{s}$:

$$\vec{r} = \vec{r}_P + \lambda \vec{s}$$



- Plane through point P normal to vector $\vec{n}$:

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_P$$

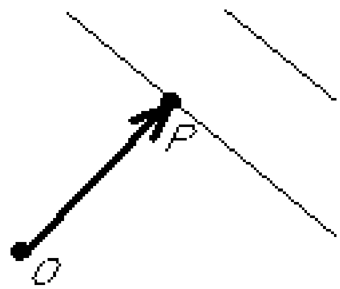


- Each equation ordinarily reduces the dimensionality by one: 3D (space) $\rightarrow$ 2D (plane) $\rightarrow$ 1D (line) $\rightarrow$ 0D (point) $\rightarrow$ nothing.

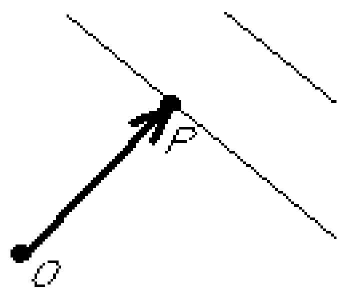
Page 477, #35(b)

1 p477, #35(b), §1 Asked

Asked: The line through point P_0, (2,-3,5), and parallel to the line $x - y + 2z + 4 = 0$, $2x + 3y + 6z - 12 = 0$.



2 p477, #35(b), §2 Identification



- I need a vector in the direction of the desired line.
- This is the same direction as the given line.
- The two equations give me vectors $\vec{n}_1$ and $\vec{n}_2$ normal to the given line
- Cross the two vectors!

3 p477, #35(b), §3 Solution

$$x - y + 2z + 4 = 0 \qquad \Longrightarrow \qquad \vec{n}_1 = (1, -1, 2)$$

$$2x + 3y + 6z - 12 = 0 \qquad \Longrightarrow \qquad \vec{n}_2 = (2, 3, 6)$$

$$\vec{s} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 3 & 6 \end{vmatrix} = \begin{pmatrix} -12 \\ -2 \\ 5 \end{pmatrix}$$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -12 \\ -2 \\ 5 \end{pmatrix}$$

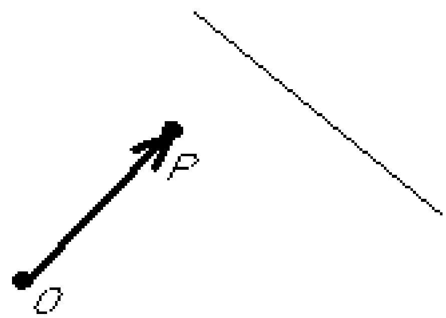
Alternatively:

$$\frac{x-2}{-12} = \frac{y+3}{-2} = \frac{z-5}{5} (= \mu)$$

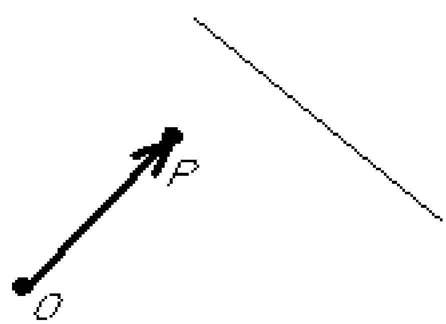
Page 477, #36(b)

1 p477, #36(b), §1 Asked

Asked: The plane through point P_0, (2,-3,2), and the line $6x+4y+3z+5=0$, $2x+y+z-2=0$.



2 p477, #36(b), §2 Identification



- I need a vector normal to the plane.
- I can get this by crossing two vectors in the plane.
- One such vector is $\vec{n}_1 \times \vec{n}_2$.
- To find another, find *any* point Q on the line, then $r_Q - r_{P_0}$ is in the plane.

3 p477, #36(b), §3 Solution

$$6x + 4y + 3z + 5 = 0 \qquad \Longrightarrow \qquad \vec{n}_1 = (6, 4, 3)$$

$$2x + y + z - 2 = 0 \qquad \Longrightarrow \qquad \vec{n}_2 = (2, 1, 1)$$

$$\vec{s} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 4 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

When $x = 0$ on the line,

$$4y + 3z + 5 = 0, \quad y + z - 2 = 0 \qquad \Longrightarrow \qquad x = 0, \quad y = -11, \quad z = 13$$

$$\vec{n} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ -11 \\ 13 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} -2 \\ -8 \\ 11 \end{pmatrix} = \begin{pmatrix} -16 \\ -7 \\ -8 \end{pmatrix}$$

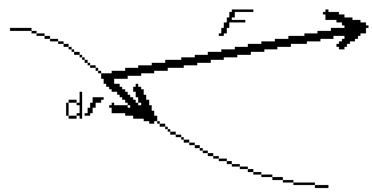
$$\begin{pmatrix} 16 \\ 7 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 7 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

$$16x + 7y + 8z = 27$$

Introduction

Line integrals:

- work;



- potential energy;
- velocity potential
- ...

Path independence:

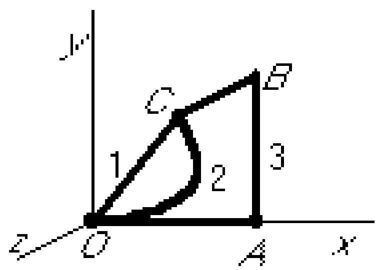
$$\int_A^B \vec{F} \cdot \mathrm{d}\vec{r}$$

is independent of the path between A and B when $\mathrm{curl}\vec{F} \equiv \mathrm{rot}\vec{F} \equiv \nabla \times \vec{F} = 0$.

Page 510, #24(a)

1 p510, #24(a), §1 Asked

Given: $\vec{F} = x\hat{\imath} + 2y\hat{\jmath} + 3x\hat{k}$



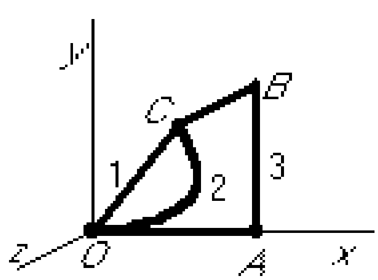
Asked: The work done by this force going from O to C along (1) the connecting line; (2) the curve $x = t$, $y = t^2$, $z = t^3$; (3) path OABC.

2 p510, #24(a), §2 Identification

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3x \end{vmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$$

3 p510, #24(a), §3 Solution

$$\int_O^C \vec{F}\,\mathrm{d}\vec{r} = \int_O^C x\,\mathrm{d}x + 2y\,\mathrm{d}y + 3x\,\mathrm{d}z$$



1. Along the line $y = x$ and $z = x$:

$$\int_{x=0}^{1} 6x \, \mathrm{d}x = 3$$

2. Along the curve $x = t$, $y = t^2$, $z = t^3$:

$$\int_{t=0}^{1} F_x \frac{\mathrm{d}x}{\mathrm{d}t} + F_y \frac{\mathrm{d}y}{\mathrm{d}t} + F_z \frac{\mathrm{d}z}{\mathrm{d}t} = \int_0^1 t \, \mathrm{d}t + 2t^2 \, 2t \, \mathrm{d}t + 3t \, 3t^2 \, \mathrm{d}t = \frac{15}{4}$$

3. Along OABC:

$$\int_{x=0}^{1} x \, \mathrm{d}x + \int_{y=0}^{1} 2y \, \mathrm{d}y + \int_{z=0}^{1} 3 \, 1 \, \mathrm{d}z = \frac{9}{2}$$

Introduction

Multiple integrals:

- Areas (cost, ...):
$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y \qquad \mathrm{d}A = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$
- Volumes (weight, ...):
$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z \qquad \mathrm{d}V = r^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta$$
- Centroids (center of gravity, center of pressure, ...)
$$\bar{x} = \int x\,\mathrm{d}A \bigg/ \int \mathrm{d}A \qquad \bar{x} = \int x\,\mathrm{d}V \bigg/ \int \mathrm{d}V$$
- Moments of inertia (solid body dynamics, center of pressure, ...)
$$I_x = \int y^2\,\mathrm{d}A \qquad I_0 = \int x^2 + y^2\,\mathrm{d}A$$
$$I_x = \int y^2 + z^2\,\mathrm{d}V \qquad I_{xy} = -\int xy\,\mathrm{d}V$$
- ...

Notes:

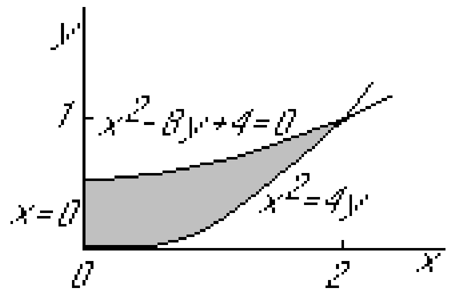
- Draw the region to be integrated over.
- When integrating, say $\int\int\int f(a,b,c)\ \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$, you have to decide whether you want to do a, b, or c first.
- Usually, you do the coordinate with the easiest limits of integration first.
- If you decide to do, say, b first, ($\int_{b_1}^{b_2} f(a,b,c)\ \mathrm{d}b$ first), the limits of integration b_1 and b_2 must be identified from the graph at *arbitrary* a and c, and are normally functions of a and c: $b_1 = b_1(a,c)$, $b_2 = b_2(a,c)$.
- After integrating over, say, b, the remaining double integral should no longer depend on b in any way. Nor does the region of integration: redraw it without the b coordinate. Then integrate over the next easiest coordinate in the same way.

Page 528, #14(e)

1 p528, #14(e), §1 Asked

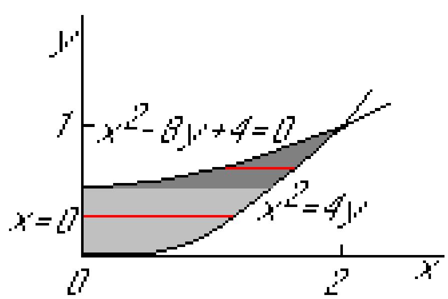
Asked: Find the centroid of the first-quadrant area bounded by $x^2 - 8y + 4 = 0$ and $x^2 = 4y$ and $x = 0$. (Slighty different from the book.)

2 p528, #14(e), §2 Region



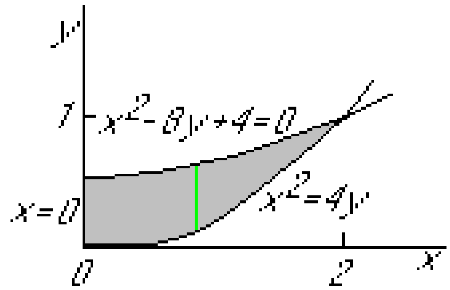
3 p528, #14(e), §3 Approach

Integrate x first?



The integral would have to be split up into the light and dark areas since the lower boundary of integration is $x = 0$ in the light region and $x = \sqrt{8y - 4}$ in the dark region.

Integrate y first!



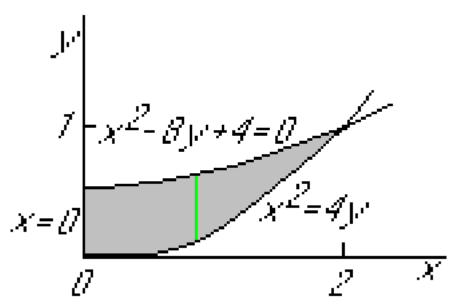
The boundaries of integration will be

$$y_1 = \frac{1}{4}x^2 \qquad y_2 = \frac{1}{8}x^2 + \frac{1}{2}$$

After integration over y, the remaining region of integration over x will be a line segment:

$$x_1 = 0 \qquad x_2 = 2$$

4 p528, #14(e), §4 Results



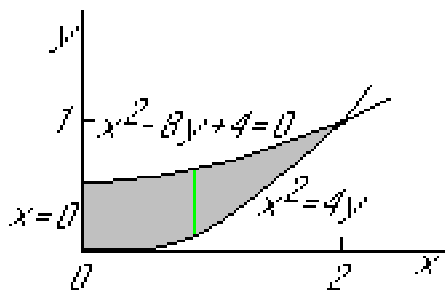
For $A = \int \mathrm{d}A = \int \int \mathrm{d}x \, \mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \mathrm{d}y \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[y\Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \, \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8}x^2 + \tfrac{1}{2}) - (\tfrac{1}{4}x^2) \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{2} - \tfrac{1}{8}x^2 \right] \mathrm{d}x = \tfrac{2}{3}$$



For $A\bar{x} = \int x \, \mathrm{d}A = \int\int x \, \mathrm{d}x \, \mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} x \;\, \mathrm{d}y \right] \, \mathrm{d}x$$

where x is constant in the integration;

$$= \int_{x=0}^{2} \left[xy\Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \, \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8}x^3 + \tfrac{1}{2}x) - (\tfrac{1}{4}x^3) \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{2}x - \tfrac{1}{8}x^3 \right] \mathrm{d}x = \tfrac{1}{2}$$

Hence $\bar{x} = \frac{1}{2}/\frac{2}{3} = \frac{3}{4}$.

For $A\bar{y} = \int y \, \mathrm{d}A = \int \int y \, \mathrm{d}x \, \mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} y \ \mathrm{d}y \right] \ \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[\frac{1}{2}y^2 \Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \ \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[\tfrac{1}{2}(\tfrac{1}{8}x^2 + \tfrac{1}{2})^2 - \tfrac{1}{2}(\tfrac{1}{4}x^2)^2 \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8} + \tfrac{1}{16}x^2 - \tfrac{3}{128}x^2 \right] \mathrm{d}x = \tfrac{4}{15}$$

Hence $\bar{x}y = \frac{4}{15}/\frac{2}{3} = \frac{2}{5}$.

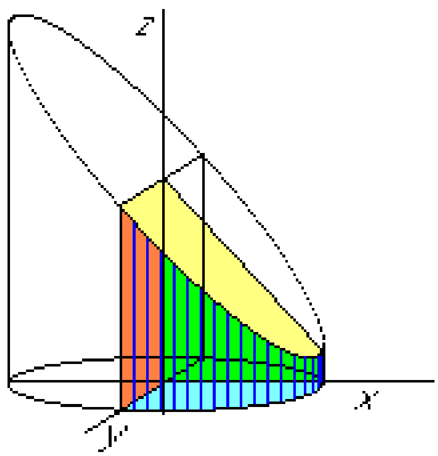
Page 549, #21(c)

1 p549, #21(c), §1 Asked

Asked: Find the centroid of the first octant region inside $x^2 + y^2 = 9$ and below $x + z = 4$.

2 p549, #21(c), §2 Approach

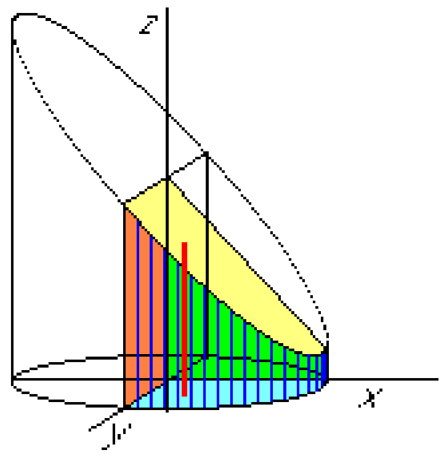
The region inside $x^2 + y^2 = 9$ is the inside of a cylinder of radius 3 around the z-axis. The equation $x + z = 4$ describes a plane through the y-axis under 45 degrees with the x-axis:



Use cylindrical coordinates r, θ, and z:

$$x = r\cos\theta \qquad y = r\sin\theta$$

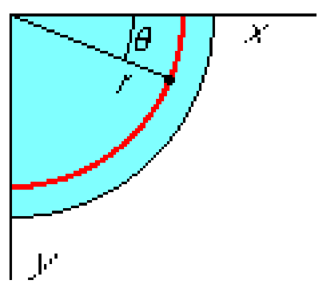
Integrate z first:



(Why not r first? Why not θ?). Boundaries are

$$z_1 = 0 \qquad z_2 = 4 - x = 4 - r\cos\theta$$

Next integrate θ and r:



$$\theta_1 = 0 \qquad \theta_2 = \tfrac{1}{2}\pi$$

$$r_1 = 0 \qquad r_2 = 3$$

3 p549, #21(c), §3 Results

For the volume $V = \int\int\int \, \mathrm{d}V = \int\int\int r \; \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$:

$$V = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} \left[\int_{z=0}^{4-r\cos\theta} r \; \mathrm{d}z \right] \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{2} (4 - r\cos\theta) r \, \mathrm{d}r \right] \, \mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2} 18 - 9\cos\theta \, \mathrm{d}\theta = 9(\pi - 1)$$

For $V\bar{x} = \int\int\int x \ \mathrm{d}V = \int\int\int xr \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$:

$$V = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} \left[\int_{z=0}^{4 - r\cos\theta} r^2 \cos\theta \ \mathrm{d}z \right] \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{2} 4r^2 \cos\theta - r^3 \cos^2\theta \, \mathrm{d}r \right] \, \mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2} 36\cos\theta - \frac{81}{4}\cos^2\theta \, \mathrm{d}\theta = \frac{9}{16}(64 - 9\pi)$$

hence $\bar{x} = (64 - 9\pi)/16(\pi - 1)$

Etcetera.