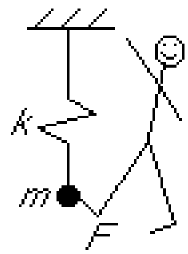
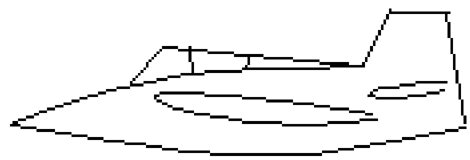
Introduction

Ordinary differential equations:

- Dynamical systems



$$m\ddot{x} + kx = F$$



$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

- Fluid mechanics
- Chemical reactions

$$\frac{\mathrm{d}O_2}{\mathrm{d}t} = -k_1[O_2][H_2] - k_2[O_2][C] + \dots$$
$$\frac{\mathrm{d}H_2}{\mathrm{d}t} = -2k_1[O_2][H_2] - k_3[H_2][C] + \dots$$

- Economics
- Biology
- ...

Notations:

- Ordinary differential equations: one independent variable
- Partial differential equations: more independent variables
- Order: order of the highest derivative
- Degree: highest degree of the dependent variable
- Linear: first degree

1.14

1 1.14, §1 Asked

Classify:

$$(y'')^2 - 3yy' + xy = 0$$

2 1.14, §2 Solution

$$(y'')^2 - 3yy' + xy = 0$$

- ordinary differential equation for $y(x)$;
- second order;
- nonlinear (second degree)

1.21

1 1.21, §1 Asked

Classify:

$$\frac{\mathrm{d}^7 b}{\mathrm{d}p^7} = 3p$$

2 1.21, §2 Solution

$$\frac{\mathrm{d}^7 b}{\mathrm{d}p^7} = 3p$$

- ordinary differential equation for $b(p)$;
- seventh order;
- linear (first degree)

Introduction

First order equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

Artificial form convenient for problems in the book, not real life:

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$$

In real life, only $f = -M/N$ would be known.

Separation of Variables

Separable equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

Solution:

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x$$

3.39

1 3.39, §1 Asked

Solve:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{t}$$

2 3.39, §2 Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{t}$$

The unknown is clearly $x(t)$.

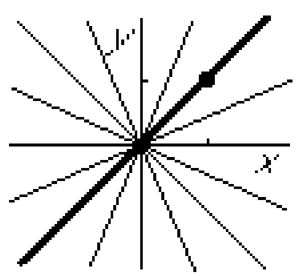
Separation of variables:

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}t}{t}$$

$$\ln|x| = \ln|t| + C$$

$$e^{\ln|x|} = e^{\ln|t|+C} \quad \Longrightarrow \quad |x| = |t|e^C \quad \Longrightarrow \quad x = \pm e^C t$$

$$x = Dt$$



An additional "initial" condition would be needed to find D. For example, $x = 1$ at $t = 1$.

Note: the O.D.E. applies at all positions. Initial or boundary conditions apply only to a specific point.

3.42

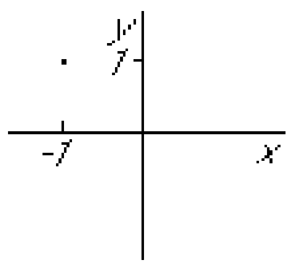
1 3.42, §1 Asked

Solve:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -(x^2+1)y \qquad y = 1 \text{ at } x = -1$$

2 3.42, §2 Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -(x^2+1)y \qquad y = 1 \text{ at } x = -1$$



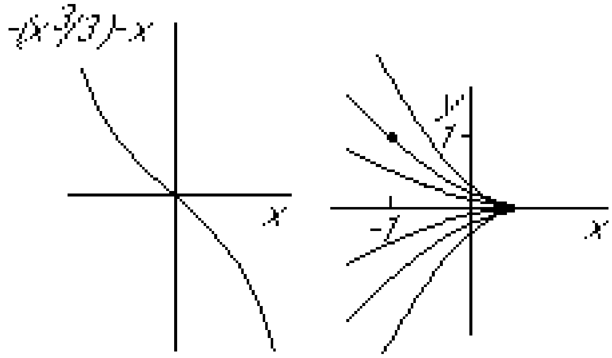
Solve the O.D.E. first:

$$\frac{\mathrm{d}y}{y} = -(x^2+1)\,\mathrm{d}x$$

$$\ln|y| = -\tfrac{1}{3}x^3 - x + C$$

$$y = \pm e^C e^{-\frac{1}{3}x^3 - x}$$

$$y = De^{-\frac{1}{3}x^3 - x}$$



Since the additional condition is $y = 1$ at $x = -1$, substitute in $y = 1$ and $x = -1$ to get D:

$$1 = De^{\frac{1}{3}+1}$$

So, at any x:

$$y = e^{-\frac{1}{3}x^3 - x - \frac{4}{3}}$$

Homogeneous Equations

Homogeneous equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$$

Solution: define a new unknown

$$u = \frac{y}{x} \ ,$$

so replace y by xu. The equation for $u(x)$ will be separable.

Note: Do not confuse this use of the word "homogeneous" for first order equations with the use of the same word for "homogeneous" for linear ODEs!

Note: To check exactness, you can replace x by tx and y by ty and see whether the ts cancel.

3.50

1 3.50, §1 Asked

Solve:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{2xy}$$

2 3.50, §2 Solution

Note that the equation is homogeneous

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{2xy} \qquad \begin{array}{l} \leftarrow \text{ degree } 2 \\ \leftarrow \text{ degree } 2 \end{array}$$

or alternatively,

$$\frac{(tx)^2 + (ty)^2}{2txty} = \frac{x^2 + y^2}{2xy}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \left(\dfrac{y}{x}\right)^2}{2\left(\dfrac{y}{x}\right)}$$

Use new unknown $u = y/x$, i.e., replace y by xu:

$$\frac{\mathrm{d}xu}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u = \frac{1 + u^2}{2u}$$

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1 - u^2}{2u}$$

Separable:

$$-\frac{2u\,\mathrm{d}u}{1 - u^2} = -\frac{\mathrm{d}x}{x}$$

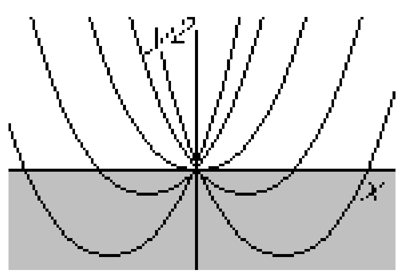
$$\ln|1 - u^2| = -\ln|x| + C$$

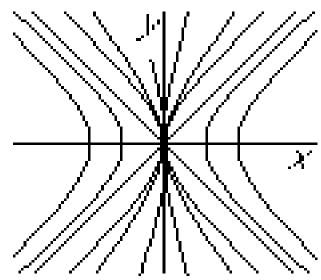
$$|1 - u^2| = \frac{e^C}{|x|}$$

$$u^2 = 1 \pm \frac{e^C}{x} = 1 + \frac{D}{x}$$

Get rid of u in favor of y/x:

$$y^2 = x^2 + Dx$$





Exact Equations

If for an equation of the form

$$g_1(x, y)\,\mathrm{d}x + g_2(x, y)\,\mathrm{d}y = 0,$$

the cross derivatives of the coefficients are equal;

$$\frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x},$$

then the equation is exact.

The solution of an exact equation is:

$$g(x, y) = C$$

where $g(x, y)$ is found by solving

$$\frac{\partial g}{\partial x} = g_1(x, y) \qquad \frac{\partial g}{\partial y} = g_2(x, y).$$

You do that by first solving the easier of the two, giving an integration constant that depends on the other variable. For example, solving $\partial g/\partial x = g_1(x, y)$ gives an integration constant depend on y. Next you take that solution and put it into the other equation.

If an equation is not exact, you may sometimes be able to find an "integrating factor" in a table.

4.32

1 4.32, §1 Asked

Solve:

$$-\frac{2y}{t^3}\,\mathrm{d}t + \frac{1}{t^2}\,\mathrm{d}y = 0$$

2 4.32, §2 Solution

$$-\frac{2y}{t^3}\,\mathrm{d}t + \frac{1}{t^2}\,\mathrm{d}y = 0$$

Check for exactness:

$$\frac{\partial g}{\partial t} \overset{?}{=} -\frac{2y}{t^3} \qquad \frac{\partial g}{\partial y} \overset{?}{=} \frac{1}{t^2}$$

$$\frac{\partial}{\partial y}\left(-\frac{2y}{t^3}\right) \overset{?}{=} \frac{\partial}{\partial t}\left(\frac{1}{t^2}\right)$$

$$-\frac{2}{t^3} \overset{!}{=} -\frac{2}{t^3}$$

Integrate the easiest equation first:

$$\frac{\partial g}{\partial y} = \frac{1}{t^2} \quad \Longrightarrow \quad g = \frac{y}{t^2} + C(t)$$

Put in the other equation:

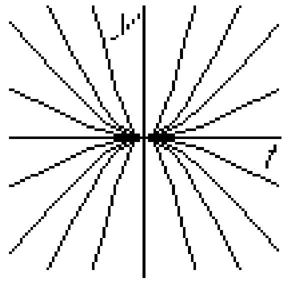
$$\frac{\partial g}{\partial t} = -\frac{2y}{t^3} + C' = -\frac{2y}{t^3}$$

$$g = \frac{y}{t^2} + C$$

Solution of the O.D.E.:

$$\frac{y}{t^2} + C = C_2$$

$$y = Dt^2$$



In real life, you would have

$$-\frac{2y}{t}\,\mathrm{d}t + \,\mathrm{d}y = 0$$

Linear Equations

Linear equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$$

The terms linear in y are the homogeneous part, the terms independent of y are the inhomogeneous terms.

Linear equations allow solutions to be added:

$$\left.\begin{array}{r} y_1' + p(x)y_1 = q_1(x) \\ y_2' + p(x)y_2 = q_2(x) \end{array}\right\}$$

$$\Longrightarrow \qquad (y_1 + y_2)' + p(x)\,(y_1 + y_2) = q_1(x) + q_2(x)$$

Solve the homogeneous equation first:

$$y' + py = 0$$

Separable:

$$\frac{\mathrm{d}y}{y} = -p\,\mathrm{d}x$$

$$y = Ce^{-\int p\,\mathrm{d}x}$$

Now solve the inhomogeneous equation:

Variation of parameter:

$$y = C(x)e^{-\int p\,\mathrm{d}x}$$

Put in the O.D.E. and solve for $C(x)$.

5.34

1 5.34, §1 Asked

Solve:

$$y' + x^2 y = x^2$$

2 5.34, §2 Solution

$$y' + x^2 y = x^2$$

The equation is linear.

Solution of the homogeneous equation:

$$y' + x^2 y = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}y}{y} = -x^2 \,\mathrm{d}x$$

$$\ln|y| = -\tfrac{1}{3}x^3 + C_1 \quad \Longrightarrow \quad y = Ce^{-\frac{1}{3}x^3}$$

Solution of the inhomogeneous equation:

$$y = C(x)e^{-\frac{1}{3}x^3}$$

into

$$y' + x^2 y = x^2$$

$$C'e^{-\frac{1}{3}x^3} - Ce^{-\frac{1}{3}x^3}x^2 + x^2Ce^{-\frac{1}{3}x^3} = x^2$$

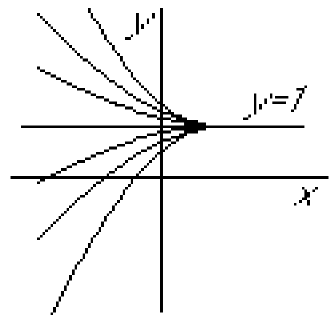
$$C' = x^2 e^{\frac{1}{3}x^3} \quad \Longrightarrow \quad C = e^{\frac{1}{3}x^3} + C_0$$

Solution:

$$y = C(x)e^{-\frac{1}{3}x^3} = 1 + C_0 e^{-\frac{1}{3}x^3}$$

Note: function $y(x) = 1$ is called a particular solution. It is *one* solution that satisfies the inhomogeneous equation.

The general solution of linear equations is always: (any arbitrary particular solution) plus (the general solution of the homogeneous equation).



(What is wrong in the graph above)?

Bernoulli Equations

Bernoulli equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n \qquad (n \neq 0, 1)$$

Solution:

Take y^n to the other side:

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y^{1-n} = q(x) \qquad (n \neq 0, 1)$$

Putting $u = y^{1-n}$ gives a linear equation:

$$\frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x} + p(x)u = q(x)$$

5.38

1 5.38, §1 Asked

Solve:

$$xy' + y = xy^3$$

2 5.38, §2 Solution

$$xy' + y = xy^3$$

It is a Bernoulli equation since it has terms linear in y and a power of y.

$$xy^{-3}y' + y^{-2} = x$$

Put $u = y^{-2}$:

$$-\tfrac{1}{2}xu' + u = x$$

Solution of the homogeneous equation:

$$-\tfrac{1}{2}xu' + u = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}u}{u} = 2\frac{\mathrm{d}x}{x} \quad \Longrightarrow \quad u = Cx^2$$

Solution of the inhomogeneous equation:

$$u = C(x)x^2$$

into the inhomogeneous equation:

$$-\tfrac{1}{2}xC'x^2 - \tfrac{1}{2}xC2x + Cx^2 = x$$

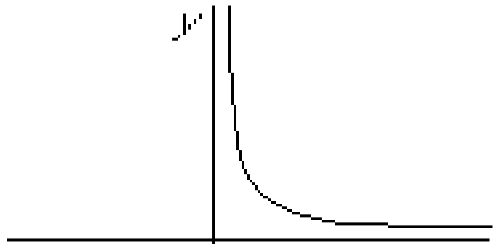
$$C' = -\frac{2}{x^2} \quad \Longrightarrow \quad C = \frac{2}{x} + C_0$$

$$u = C(x)x^2 = 2x + C_0x^2 = \frac{1}{y^2}$$

Solution:

$$y = \frac{\pm 1}{\sqrt{2x + C_0x^2}}$$

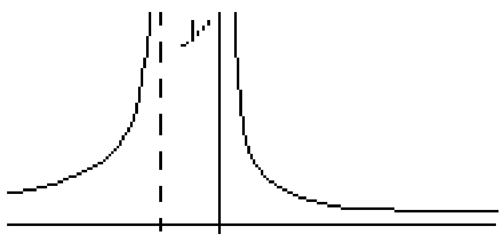
For $C_0 = 0$ $y = \pm 1/\sqrt{2x}$:



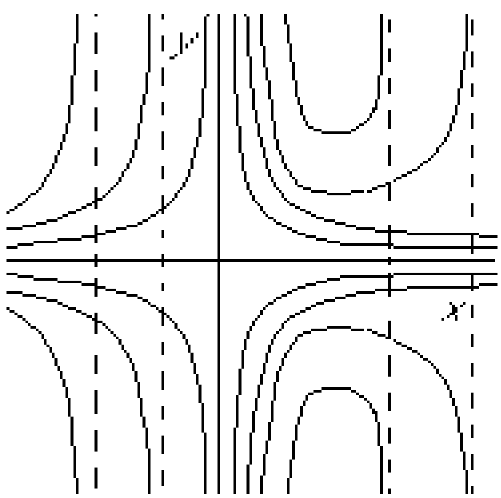
For $x = -2/C_0$, y is infinite.

For $C_0 < 0$:

For $C_0 > 0$:



Total:



Introduction

Linear Constant Coefficient Equations:

- dynamical systems;
- vibrating systems;
- linearized systems;
- part of the solution of multidimensional problems;
- ...

General form:

$$a_0 y + a_1 y' + a_2 y'' + a_3 y^{(3)} + \ldots + a_n y^{(n)} = q$$

where $a_0, a_1, \ldots, a_n$ are all constants but q can be any function of x.

Solution of the homogeneous equation:

Homogeneous equation:

$$a_0 y + a_1 y' + a_2 y'' + a_3 y^{(3)} + \ldots + a_n y^{(n)} = 0$$

Special solutions are $y = e^{\lambda x}$ provided that λ is a root of the characteristic polynomial:

$$a_0 + a_1 \lambda + a_2 \lambda^2 + a_3 \lambda^3 + \ldots + a_n \lambda^n = 0$$

If all roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are different, the *general* solution of the homogeneous equation is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \ldots + C_n e^{\lambda_n x}$$

8.18

1 8.18, §1 Asked

Solve:

$$y'' - y' - 30y = 0$$

2 8.18, §2 Solution

$$y'' - y' - 30y = 0$$

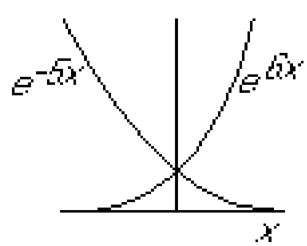
Characteristic polynomial:

$$\lambda^2 - \lambda - 30 = 0$$

has roots $\lambda_1 = 6$ and $\lambda_2 = -5$

General solution:

$$y = C_1 e^{6x} + C_2 e^{-5x}$$



8.19

1 8.19, §1 Asked

Solve:

$$y'' - 2y' + y = 0$$

2 8.19, §2 Solution

$$y'' - 2y' + y = 0$$

Characteristic polynomial:

$$\lambda^2 - 2\lambda + 1 = 0$$

has roots $\lambda_1 = \lambda_2 = 1$.

For multiple roots, start adding factors that are increasing powers of x: General solution:

$$y = C_1 e^x + C_2 x e^x$$

8.21

1 8.21, §1 Asked

Solve:

$$y'' + 2y' + 2y = 0$$

2 8.21, §2 Solution

$$y'' + 2y' + 2y = 0$$

Characteristic polynomial:

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i \qquad (i = \sqrt{-1})$$

General solution:

$$y = C_1 e^{(-1+i)x} + C_2 e^{(-1-i)x}$$

Cleanup of complex exponentials is required in this class:

$$y = e^{-x}\left(C_1 e^{ix} + C_2 e^{-ix}\right)$$

Euler:

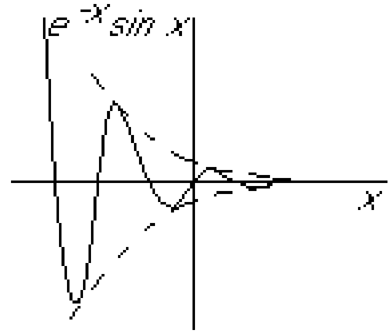
$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

$$y = e^{-x}\left(C_1[\cos x + i\sin x] + C_2[\cos x - i\sin x]\right)$$

$$y = e^{-x}\left([C_1 + C_2]\cos x + i[C_1 - C_2]\sin x\right)$$

Cleaned up solution:

$$y = e^{-x}\left(D_1 \cos x + D_2 \sin x\right)$$



$e^{-x}\sin x$
x

Method of Undetermined Coefficients

Inhomogeneous equation:

$$a_0 y + a_1 y' + a_2 y'' + a_3 y^{(3)} + \ldots + a_n y^{(n)} = q$$

where $q \neq 0$.

First solve the homogeneous equation, then guess a particular solution with a few undetermined coefficients:

For $q =$:	guess $y_p =$:
$e^{\alpha x}$	$Ce^{\alpha x}$
$e^{\lambda x}$	$Cx^n e^{\lambda x}$
$\cos x$	$C_1 \cos x + C_2 \sin x$
polynomial	polynomial
$\ldots$	$\ldots$

The general solution is any particular solution plus the general solution of the homogeneous equation.

10.45

1 10.45, §1 Asked

Solve:

$$y'' - 2y' + y = 3e^{2x}$$

2 10.45, §2 Solution

$$y'' - 2y' + y = 3e^{2x}$$

Homogeneous equation:

Characteristic polynomial:

$$\lambda^2 - 2\lambda + 1 = 0$$

has roots $\lambda_1 = \lambda_2 = 1$: General solution:

$$y_h = C_1 e^x + C_2 x e^x$$

Particular solution:

$$y_p'' - 2y_p' + y_p = 3e^{2x}$$

Guessing $y_p = Ce^{2x}$ produces

$$y_p'' - 2y_p' + y_p = C\left(4e^{2x} - 4e^{2x} + e^{2x}\right) = Ce^{2x}$$

so $C = 3$ and $y_p = 3e^{2x}$.

Total solution:

$$y = 3e^{2x} + C_1 e^x + C_2 x e^x$$

10.47

1 10.47, §1 Asked

Solve:

$$y'' - 2y' + y = 3e^x$$

2 10.47, §2 Solution

$$y'' - 2y' + y = 3e^x$$

Homogeneous equation:

$$y_h = C_1 e^x + C_2 x e^x$$

Particular solution:

$$y_p'' - 2y_p' + y_p = 3e^x$$

Particular solutions $y_p = e^x$ or $y_p = xe^x$ don't work. Try

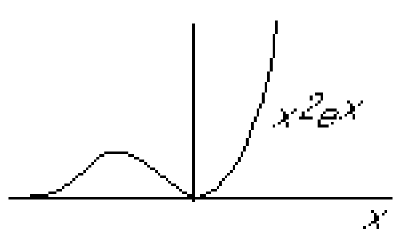
$$y_p = Ce^x x^2 \quad y_p' = Ce^x(x^2 + 2x) \quad y_p'' = Ce^x(x^2 + 4x + 2)$$

$$y_p'' - 2y_p' + y_p = Ce^x 2$$

so $C = 1.5$.

Total solution:

$$y = 1.5x^2 e^x + C_1 e^x + C_2 x e^x$$



Variation of Parameters

Works always, but is more work.

Inhomogeneous equation:

$$a_0 y + a_1 y' + a_2 y'' + a_3 y^{(3)} + \ldots + a_n y^{(n)} = q$$

where $q \neq 0$.

First solve the homogeneous equation, then allow its integration constants to vary.

11.10

1 11.10, §1 Asked

Solve:

$$y'' + y = \sec x$$

2 11.10, §2 Solution

$$y'' + y = \sec x$$

Homogeneous equation:

$$\lambda^2 + 1 = 0 \implies \lambda = \pm i$$

$$y_h = A\cos x + B\sin x$$

Variation of parameters:

$$y = A\cos x + B\sin x \qquad A = A(x), B = B(x)$$

$$y' = -A\sin x + B\cos x + A'\cos x + B'\sin x$$

Put the additional terms to zero:

$$A'\cos x + B'\sin x = 0 \tag{1}$$

$$y'' = -A\cos x - B\sin x - A'\sin x + B'\cos x$$

Do not put the additional terms to zero in the highest derivative. Instead, put everything into the O.D.E.:

$$y'' + y = -A\cos x - B\sin x - A'\sin x + B'\cos x + A\cos x + B\sin x = \sec x$$

$$-A'\sin x + B'\cos x = \sec x \tag{2}$$

The result is a system of linear equations (1), (2) for A' and B':

$$\left(\begin{array}{cc|c} \cos x & \sin x & 0 \\ -\sin x & \cos x & \sec x \end{array}\right) \qquad \begin{array}{l} (1) \\ (2) \end{array}$$

Forward elimination:

$$\left(\begin{array}{cc|c} \cos x & \sin x & 0 \\ 0 & 1 & 1 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') = \cos x(2) + \sin x(1) \end{array}$$

Back substitution gives $B' = 1$ and $A' = -\tan x$:

$$B = x + B_0 \qquad A = \ln|\cos x| + A_0$$

Total solution $y = A\cos x + B\sin x$:

$$y = \ln|\cos x|\cos x + x\sin x + A_0\cos x + B_0\sin x$$

11.25

1 11.25, §1 Asked

Solve:

$$\dddot{r} - 3\ddot{r} + 3\dot{r} - r = \frac{e^t}{t}$$

2 11.25, §2 Solution

$$\dddot{r} - 3\ddot{r} + 3\dot{r} - r = \frac{e^t}{t}$$

Homogeneous equation:

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \quad \Longrightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = 1$$
$$r_h = C_1 e^t + C_2 t e^t + C_3 t^2 e^t$$

Variation of parameters:

$$r = C_1 e^t + C_2 t e^t + C_3 t^2 e^t$$
$$\dot{C}_1 e^t + \dot{C}_2 t e^t + \dot{C}_3 t^2 e^t = 0 \tag{1}$$
$$\dot{r} = C_1 e^t + C_2 (t+1) e^t + C_3 (t^2 + 2t) e^t$$
$$\dot{C}_1 e^t + \dot{C}_2 (t+1) e^t + \dot{C}_3 (t^2 + 2t) e^t = 0 \tag{2}$$
$$\ddot{r} = C_1 e^t + C_2 (t+2) e^t + C_3 (t^2 + 4t + 2) e^t$$
$$\dddot{r} = \dot{C}_1 e^t + \dot{C}_2 (t+2) e^t + \dot{C}_3 (t^2 + 4t + 2) e^t + \ldots$$

Into the O.D.E.:

$$\dot{C}_1 e^t + \dot{C}_2 (t+2) e^t + \dot{C}_3 (t^2 + 4t + 2) e^t + \ldots = \frac{e^t}{t} \tag{3}$$

Total system of equations for unknowns $\dot{C}_1$,$\dot{C}_2$, and $\dot{C}_3$:

$$\left(\begin{array}{ccc|c} 1 & t & t^2 & 0 \\ 1 & t+1 & t^2 + 2t & 0 \\ t & t^2 + 2t & t^3 + 4t^2 + 2t & 1 \end{array}\right) \qquad \begin{array}{l} (1') = e^{-t}(1) \\ (2') = e^{-t}(2) \\ (3') = t e^{-t}(3) \end{array}$$

Forward elimination:

$$\left(\begin{array}{ccc|c} 1 & t & t^2 & 0 \\ 0 & 1 & 2t & 0 \\ 0 & 2t & 4t^2+2t & 1 \end{array}\right) \qquad \begin{array}{l} (1') \\ (2'') = (2') - (1') \\ (3'') = (3') - t(1') \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & t & t^2 & 0 \\ 0 & 1 & 2t & 0 \\ 0 & 0 & 2t & 1 \end{array}\right) \qquad \begin{array}{l} (1') \\ (2'') \\ (3''') = (3'') - 2t(2'') \end{array}$$

Back substitution: $\dot{C}_3 = 1/2t$, $\dot{C}_2 = -1$, $\dot{C}_1 = \frac{1}{2}t$, hence

$$C_1 = \tfrac{1}{4}t^2 + C_{10} \quad C_2 = -t + C_{20} \quad C_3 = \ln\sqrt{t} + C_{30}$$

Total solution:

$$r = t^2 \ln\sqrt{t} e^t + C_{10}e^t + C_{20}te^t + C_{30}^* t^2 e^t$$

Initial Conditions

After solving the O.D.E., a finite number of unknown integration constants remain. In practical applications, these integration constants are typically found from *initial conditions* at a starting point such as $t = 0$ or from *boundary conditions* at the end points a and b of an interval $a \leq x \leq b$

12.11

1 12.11, §1 Asked

Solve:

$$y'' + y = x \qquad y(1) = 0, y'(1) = 1$$

2 12.11, §2 Solution

$$y'' + y = x \qquad y(1) = 0, y'(1) = 1$$

Homogeneous solution:

$$y_h = A\cos x + B\sin x$$

Guess the particular solution $Cx + D$:

$$y_p = x$$

General solution:

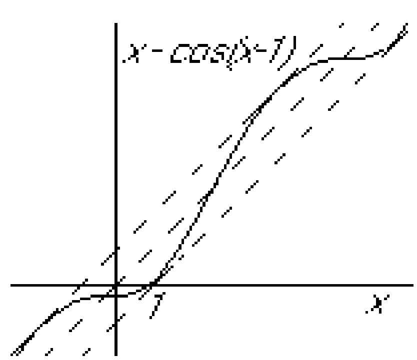
$$y = x + A\cos x + B\sin x$$

Put in the initial conditions

$$y(1) = 1 + A\cos 1 + B\sin 1 = 0 \quad y'(1) = 1 - A\sin 1 + B\cos 1 = 1$$

to find $A = -\cos 1$ and $B = -\sin 1$:

$$y = x - \cos 1\cos x - \sin 1\sin x = x - \cos(x-1)$$



First Order Systems

Important for numerical work. Library subroutines usually do not solve higher order equations, but they do solve first order systems.

General First Order System:

$$\vec{y}' = \vec{f}(x, \vec{y})$$

Written out

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2, \ldots, y_n) \\ y_2' &= f_2(x, y_1, y_2, \ldots, y_n) \\ &\cdots \\ y_n' &= f_n(x, y_1, y_2, \ldots, y_n) \end{aligned}$$

If the functions are linear constant coefficient ones, we can rewrite this as:

$$\vec{y}' = A\vec{y} + b(x).$$

In this class, solution using eigenvalues and eigenvectors is *required.* We assume that A is diagonalizable.

Homogeneous solution:

$$y_h = C_1 \vec{v}_1 e^{\lambda_1 x} + C_2 \vec{v}_2 e^{\lambda_2 x} + \ldots$$

where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of A and $\vec{v}_1, \vec{v}_2, \ldots$ the eigenvectors.

General solution: Guess and add a particular solution. Varying the parameters C_1, C_2, ... also works.

22.12

1 22.12, §1 Asked

Solve as a System:

$$\ddot{x} + 2\dot{x} - 8x = 4 \qquad x(0) = 1, \dot{x}(0) = 2$$

2 22.12, §2 Solution

Solve as a System:

$$\ddot{x} + 2\dot{x} - 8x = 4 \qquad x(0) = 1, \dot{x}(0) = 2$$

Define new dependent variables $x_1 = x$ and $x_2 = \dot{x}$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 8x_1 - 2x_2 + 4 \end{aligned}$$

Matrix form $\dot{\vec{x}} = A\vec{x} + b$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Homogeneous equation:

$$\vec{x}_h = C_1\vec{v}_1 e^{\lambda_1 t} + C_2\vec{v}_2 e^{\lambda_2 t}$$

where λ_1 and λ_2 are the eigenvalues of A and $\vec{v}_1$ and $\vec{v}_2$ the eigenvectors:

$$\begin{vmatrix} -\lambda & 1 \\ 8 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 8 = 0$$

$$\lambda_1 = 2, \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \lambda_2 = -4, \vec{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Particular solution $\dot{\vec{x}}_p = A\vec{x}_p + b$: guess that $\vec{x}_p$ is constant, then $A\vec{x}_p = -\vec{b}$. Solve:

$$x_p = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$$

Total solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t}$$

Put in the initial conditions:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} C_1 \\ 2C_1 \end{pmatrix} + \begin{pmatrix} C_2 \\ -4C_2 \end{pmatrix}$$

which gives $C_2 = \frac{1}{6}$, $C_1 = \frac{4}{3}$.

Final solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{4}{3} \\ \frac{8}{3} \end{pmatrix} e^{2t} + \begin{pmatrix} \frac{1}{6} \\ -\frac{2}{3} \end{pmatrix} e^{-4t}$$

Some Other Equations

1 §1 Introduction

Generally speaking, equations become more difficult when the order goes up.

For a first order equation, even if you cannot solve, you can always draw little line segments with the right slope in the x, y plane and then draw trajectories following those directions.

For some equations there are tricks that allow you to reduce the order.

Nonlinear equations are generally more difficult than linear ones.

2 §2 Handbooks

Look it up in a mathematical handbook. Schaum's Mathematical Handbook has some. Abramowitz and Stegun has a large collection of equations solvable by Bessel functions and other standard functions, and the properties of these function.

Avoid exact equations in Schaum's Mathematical Handbook.

3 §3 Power Series

Expand the solution in a power series, equating all powers in the O.D.E. to zero.

4 §4 Euler

$$a_0 y + a_1 x y' + a_2 x^2 y'' + a_3 x^3 y^{(3)} + \ldots + a_n x^n y^{(n)} = q$$

where $a_0, a_1, \ldots, a_n$ are all constants but q can be any function of x.

The substitution $\xi = \ln x$ turns this into a constant coefficient equation for $y(\xi)$. Reason:

$$y' = \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}y = \frac{1}{x}y_\xi$$

$$y'' = -\frac{1}{x^2}y_\xi + \frac{1}{x}\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}y_\xi = -\frac{1}{x^2}y_\xi + \frac{1}{x^2}y_{\xi\xi}$$

$$y''' = \ldots$$

5 §5 No y

If the derivatives of y, but not y itself appear, simply take y' instead of y as the unknown. A second order equation for y becomes first order for y'.

6 §6 No x

If derivatives with respect to x appear, but not x itself, use y as the new *independent* variable and y' as the new dependent variable.

$$y'' = \frac{\mathrm{d}y'}{\mathrm{d}y}y'$$

$$y''' = \ldots$$

The order of the equation for $y'(y)$ is one less than that of the equation for $y(x)$.

7 §7 Linear

If the equation is linear and homogeneous, setting $y = e^f$ gives an equation not involving f itself:

$$y = e^f \qquad y' = e^f f' \qquad y'' = e^f f'^2 + e^f f'' \qquad \ldots$$

If the equation is linear and homogeneous, and you know a solution $y_1(x)$, setting $y = C(x)y_1(x)$ gives a *linear* equation for C not involving C itself.