11.10

1 11.10, §1 Asked

Solve:

$$y'' + y = \sec x$$

2 11.10, §2 Solution

$$y'' + y = \sec x$$

Homogeneous equation:

$$\lambda^2 + 1 = 0 \quad \Longrightarrow \quad \lambda = \pm i$$

$$y_h = A\cos x + B\sin x$$

Variation of parameters:

$$y = A\cos x + B\sin x \qquad A = A(x), B = B(x)$$

$$y' = -A\sin x + B\cos x + A'\cos x + B'\sin x$$

Put the additional terms to zero:

$$A'\cos x + B'\sin x = 0 \tag{1}$$

$$y'' = -A\cos x - B\sin x - A'\sin x + B'\cos x$$

Do not put the additional terms to zero in the highest derivative. Instead, put everything into the O.D.E.:

$$y'' + y = -A\cos x - B\sin x - A'\sin x + B'\cos x + A\cos x + B\sin x = \sec x$$

$$-A'\sin x + B'\cos x = \sec x \tag{2}$$

The result is a system of linear equations (1), (2) for A' and B':

$$\left(\begin{array}{cc|c} \cos x & \sin x & 0 \\ -\sin x & \cos x & \sec x \end{array}\right) \qquad \begin{array}{l} (1) \\ (2) \end{array}$$

Forward elimination:

$$\left(\begin{array}{cc|c} \cos x & \sin x & 0 \\ 0 & 1 & 1 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') = \cos x(2) + \sin x(1) \end{array}$$

Back substitution gives $B' = 1$ and $A' = -\tan x$:

$$B = x + B_0 \qquad A = \ln|\cos x| + A_0$$

Total solution $y = A\cos x + B\sin x$:

$$y = \ln|\cos x|\cos x + x\sin x + A_0\cos x + B_0\sin x$$