3.57(a)

1 3.57(a), §1 Asked

Given:

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \vec{u}_2 = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \vec{u}_3 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

Asked: Express

$$\vec{v} = \begin{pmatrix} 4 \\ -9 \\ 2 \end{pmatrix}$$

in terms of $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$.

2 3.57(a), §2 Solution

We need c_1, c_2, and c_3 so that

$$\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$$

In matrix form:

$$\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \vec{v}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 4 & -3 & -9 \\ -1 & 2 & 2 & 2 \end{array}\right) \qquad \begin{pmatrix} (1) \\ (2) \\ (3) \end{pmatrix}$$

Forward elimination:

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 4 \\ 0 & \boxed{2} & -5 & -17 \\ 0 & 3 & 3 & 6 \end{array}\right) \qquad \begin{pmatrix} (1) \\ (2') = (2) - 2(1) \\ (3') = (3) + (1) \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 4 \\ 0 & \boxed{2} & -5 & -17 \\ 0 & 0 & \boxed{21} & 63 \end{array}\right) \qquad \begin{pmatrix} (1) \\ (2') \\ (3'') = 2(3') - 3(2') \end{pmatrix}$$

Back substitution:

From ($3''$), $c_3 = 3$; from ($2'$), $c_2 = -1$; from (1), $c_1 = 2$.

If the right hand side $\vec{v}$ would have been zero, the only possible values for c_1, c_2, and c_3 would be all zero. A set of vectors is dependent if you can create zero from them with some nonzero coefficients. (This allows you to express one of the set in terms of the others.)

Since you cannot do so with u_1, u_2 and u_3, they are independent vectors.

Also, since you can find a solution for any vector $\vec{v}$, you can express *any* vector in terms of u_1, u_2, and u_3. Vectors for which that is true are called a basis, in this case for three-dimensional vector space.