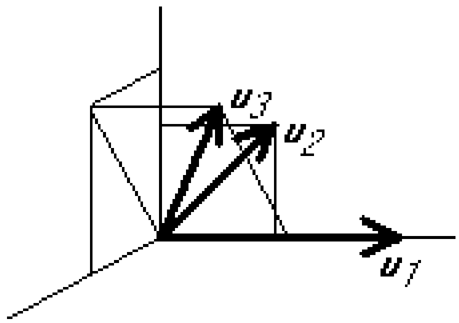
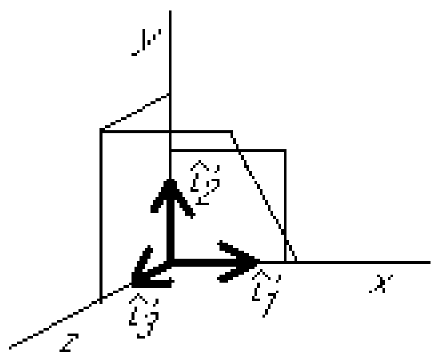
Gram-Schmidt

Description:

Gram-Schmidt orthogonalization is a way of converting a given arbitrary basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ into an equivalent orthonormal basis:





This often leads to better accuracy (e.g. in least square problems) and/or simplifications.

Modified Gram-Schmidt Procedure

Given a set of linearly independent vectors, $\vec{u}_1, \vec{u}_2, \ldots$, turn them into an equivalent orthonormal set $\hat{\imath}'_1, \hat{\imath}'_2, \ldots$ as follows:

Step 1:

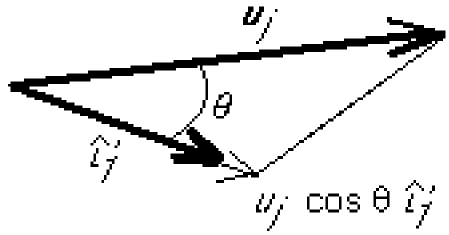
1. Normalize the first vector $\vec{u}_1$. That will be your $\hat{\imath}'_1$

$$\hat{\imath}'_1 = \frac{\vec{u}_1}{||\vec{u}_1||}$$

2. For the remaining vectors $\vec{u}_2, \vec{u}_3, \ldots$, eliminate their component in the direction of $\hat{\imath}'_1$ using the following formula:

$$\vec{u}^*_j = \vec{u}_j - \hat{\imath}'_1 \left(\hat{\imath}'^H_1 \vec{u}_j \right)$$

Note that $\hat{\imath}_1'^H \vec{u}_j = ||\hat{\imath}_1'|| ||\vec{u}_j|| \cos\theta = ||\vec{u}_j|| \cos\theta$ is the component of $\vec{u}_j$ in the direction of $\hat{\imath}_1'$:



Also $\hat{\imath}_1' \hat{\imath}_1'^H \vec{u}_j = \mathrm{proj}(\hat{\imath}_1', \vec{u}_j)$. The matrix $\hat{\imath}_1' \hat{\imath}_1'^H$ is called the projection operator onto $\hat{\imath}_1'$.

Ignore $\hat{\imath}_1'$ in the remaining process.

Step 2:

1. Normalize the second vector $\vec{u}_2^*$. That will be your $\hat{\imath}_2'$

$$\hat{\imath}_2' = \frac{\vec{u}_2^*}{||\vec{u}_2^*||}$$

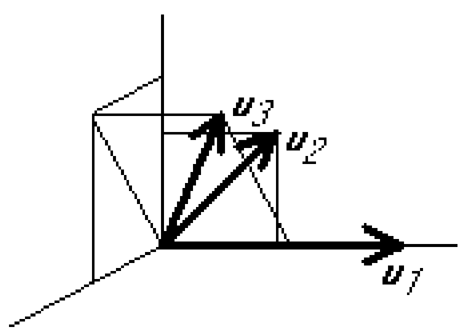
2. For the remaining vectors $\vec{u}_3, \vec{u}_4, \ldots$, eliminate their component in the direction of $\hat{\imath}_2'$ using the following formula:

$$\vec{u}_j^{**} = \vec{u}_j^* - \hat{\imath}_2' \left(\hat{\imath}_2'^H \vec{u}_j^* \right)$$

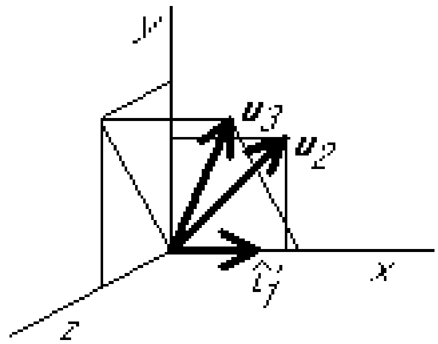
Ignore $\hat{\imath}_2'$ in the remaining process.

Repeat the process along the same lines until you run out of vectors.

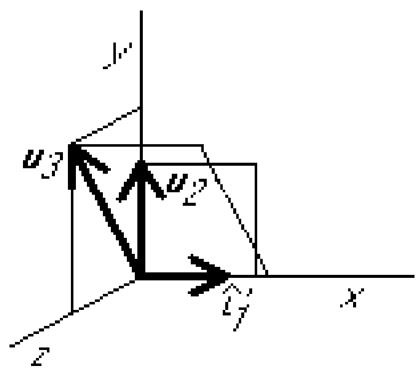
Graphical example:



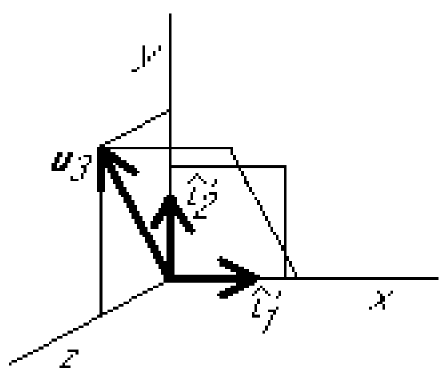
Normalize $\vec{u}_1$:



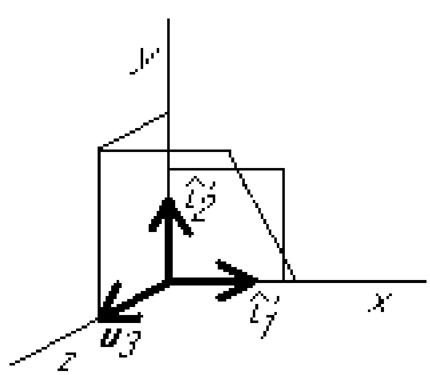
Eliminate the components in the $\vec{u}_1$ direction from the rest:



Normalize $\vec{u}_2$:



Eliminate the components in the $\vec{u}_2$ direction from the rest:



Normalize $\vec{u}_3$: