7.21

1 7.21, §1 Asked

Given: The basis vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix}$$

Asked: Find (a) an orthogonal basis for the space spanned by $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$; (b) an orthonormal basis for the space spanned by $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

2 7.21, §2 Solution

Since an orthonormal basis *is* orthogonal, I only need do (b).

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix}$$

Normalize v_1:

$$\hat{\imath}'_1 = \frac{\vec{v}_1}{||\vec{v}_1||} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Get rid of the $\hat{\imath}'_1$-components:

$$\vec{v}_2^* = \vec{v}_2 - \hat{\imath}'_1 \hat{\imath}'^H_1 \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} 4 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

$$\vec{v}_3^* = \vec{v}_3 - \hat{i}_1'\hat{i}_1'^H\vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -4 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} (-2) = \begin{pmatrix} 2 \\ 3 \\ -3 \\ -2 \end{pmatrix}$$

Normalize v_2^*:

$$\hat{i}_2' = \frac{\vec{v}_2^*}{||\vec{v}_2^*||} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

Get rid of the $\hat{i}_2'$-components:

$$\vec{v}_3^{**} = \vec{v}_3^* - \hat{i}_2'\hat{i}_2'^H\vec{v}_3^* = \begin{pmatrix} 2 \\ 3 \\ -3 \\ -2 \end{pmatrix} - \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -3 \\ -2 \end{pmatrix} - \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \end{pmatrix} \frac{-9}{\sqrt{6}} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -3 \\ 1 \end{pmatrix}$$

Normalize v_3^{**}:

$$\hat{i}_3' = \frac{\vec{v}_3^{**}}{||\vec{v}_3^{**}||} = \begin{pmatrix} \frac{1}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} \\ -\frac{6}{5\sqrt{2}} \\ \frac{2}{5\sqrt{2}} \end{pmatrix}$$