Transpose matrices

1 General

Transposing a matrix turns the columns into rows and vice-versa

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{m2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{mn} \end{pmatrix}$$

Similarly, transposing turns a column vector into a row vector and vice-versa.

Another way of thinking about it is that the elements are flipped over around the *"main diagonal"*, which runs from top left to bottom right:

$$\begin{pmatrix} \boxed{a_{11}} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & \boxed{a_{22}} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & \boxed{a_{33}} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & \boxed{a_{nn}} \end{pmatrix}$$

(The sum of the elements on the main diagonal is called the *trace* of the matrix.)

Note that $\left(A^T\right)^T = A$.

Transpose in index notation:

$$a_{ij}^T = a_{ji} \text{ for all } i \text{ and } j$$

Note that in index notation, the main diagonal consists of the elements where $i = j$. These stay put during transposing.

Transposing matrix products:

$$(AB)^T = B^T A^T$$

For complex matrices, the normal generalization of transpose is "Hermitian conjugate", where you take the complex conjugate of each complex number, *in addition* to interchanging rows and columns: $A^H \equiv \bar{A}^T$, or $a_{ij}^H = \bar{a}_{ji}$.

Example:

$$\begin{pmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{pmatrix}^H = \begin{pmatrix} 1-2i & 5-6i \\ 3-4i & 7-8i \end{pmatrix}$$

2 Special matrices

Symmetric matrices satisfy

$$S^T = S$$

Symmetric matrices are very common in engineering. For example, most statics deals with symmetric matrices, as does solid body dynamics, and a lot of the simpler fluid flows.

Complex matrices for which $A^H = A$ are called "Hermitian matrices." They are all over quantum mechanics.

Skew-symmetric matrices satisfy

$$K^T = -K$$

Skew-symmetric matrices determine the velocity field in solid body motion, and other fields involving cross products.

Example: the following is a skew symmetric matrix:

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$

Diagonal matrices have only nonzero elements on the main diagonal:

$$D = \begin{pmatrix} d_{11} & 0 & 0 & \dots & 0 \\ 0 & d_{22} & 0 & \dots & 0 \\ 0 & 0 & d_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{nn} \end{pmatrix}$$

An example is the unit matrix. In index notation, a matrix is diagonal iff $d_{ij} = 0$ if $i \neq j$.

Upper triangular matrices have only nonzero elements on and above the main diagonal:

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{pmatrix}$$

In index notation, $u_{ij} = 0$ if $j < i$.

Lower triangular matrices:

$$L = \begin{pmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ l_{31} & l_{32} & l_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \dots & l_{nn} \end{pmatrix}$$

In index notation, $l_{ij} = 0$ if $j > i$.

The transpose of an upper triangular matrix is a lower triangular one and vice-versa.