Matrix multiplication

1 General

Matrix multiplication is defined in terms of the *row-column* product:

$$C = AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1p} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ip} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & \dots & b_{1n} \\ b_{21} & \dots & b_{2j} & \dots & \dots & b_{2n} \\ b_{31} & \dots & b_{3j} & \dots & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{p1} & \dots & b_{pj} & \dots & \dots & b_{pn} \end{pmatrix}$$

$$C = \begin{pmatrix} c_{11} & \dots & \dots & \dots & c_{1n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \dots & \dots & c_{ij} & \dots & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ c_{m1} & \dots & \dots & \dots & c_{mn} \end{pmatrix}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}$$

In other words, c_{ij} is the dot product of the i-th *row*-vector of A times the j-th *column*-vector of B:

$$AB = \begin{pmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vec{a}_3^T \\ \vdots \\ \vec{a}_m^T \end{pmatrix} \begin{pmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \dots & \vec{b}_n \end{pmatrix} = \begin{pmatrix} \vec{a}_1^T \cdot \vec{b}_1 & \vec{a}_1^T \cdot \vec{b}_2 & \vec{a}_1^T \cdot \vec{b}_3 & \dots & \vec{a}_1^T \cdot \vec{b}_n \\ \vec{a}_2^T \cdot \vec{b}_1 & \vec{a}_2^T \cdot \vec{b}_2 & \vec{a}_2^T \cdot \vec{b}_3 & \dots & \vec{a}_2^T \cdot \vec{b}_n \\ \vec{a}_3^T \cdot \vec{b}_1 & \vec{a}_3^T \cdot \vec{b}_2 & \vec{a}_3^T \cdot \vec{b}_3 & \dots & \vec{a}_3^T \cdot \vec{b}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{a}_m^T \cdot \vec{b}_1 & \vec{a}_m^T \cdot \vec{b}_2 & \vec{a}_m^T \cdot \vec{b}_3 & \dots & \vec{a}_m^T \cdot \vec{b}_n \end{pmatrix}$$

(Here the first row of A is written as $\vec{a}_1^T$, the second row as $\vec{a}_2^T$, etc. Similar, the first column of B is $\vec{b}_1$, etc.)

The dots in the above product can be omitted since the matrix product of a row vector times a column vector is by definition the same as the dot product of those vectors.

Multiplication in index notation:

$$C = AB \qquad \Longrightarrow \qquad c_{ij} = \sum_k a_{ik}b_{kj} \text{ for all } i \text{ and } j$$

The summation is over *neighboring indices.*

For matrices to be multiplied, the second dimension of A must be the same as the first dimension of B.

Matrix multiplication does not ordinarily commute:

$$AB \neq BA$$

2 Unit matrix

The unit (or identity) matrix I is like the number 1 for numbers: multiplying by I does not change a matrix.

Form of the unit matrix:

$$I = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Note, blocks of zeros are often omitted, (or written as a humongous zero,) so

$$I = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Index notation

$$I_{ij} = \delta_{ij} \qquad (= 1 \text{ if } i = j; \quad = 0 \text{ if } i \neq j)$$

The tensor δ_{ij} is called the Kronecker delta.