Inverse Matrices

1 General

Inverse matrices are like inverses for numbers:

$$AA^{-1} = A^{-1}A = I$$

Note that $(A^{-1})^{-1} = A$.

The inverse only exists when the determinant of the matrix, $|A|$, is nonzero.

To get the inverses of *small* matrices, you could use the procedure of taking "minors". In index notation:

$$a_{ij}^{-1\,T} = (-1)^{i+j}|A_{ij}|/|A|$$

where A_{ij} is the matrix A after you remove the column and row of element a_{ij}. See the example problems.

Inverting products:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Transposing and inversing commute:

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

2 Orthonormal matrices

Orthonormal (orthogonal) matrices are matrices in which the columns vectors form an orthonormal set (each column vector has length one and is orthogonal to all the other colum vectors).

For square orthonormal matrices, the inverse is simply the transpose,

$$Q^{-1} = Q^T$$

This can be seen from:

$$Q^TQ = \begin{pmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \\ \vdots \\ \vec{q}_n^T \end{pmatrix} \begin{pmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \dots & \vec{q}_n \end{pmatrix}$$

$$
= \begin{pmatrix}
\vec{q}_1^T \vec{q}_1 & \vec{q}_1^T \vec{q}_2 & \vec{q}_1^T \vec{q}_3 & \dots & \vec{q}_1^T \vec{q}_n \\
\vec{q}_2^T \vec{q}_1 & \vec{q}_2^T \vec{q}_2 & \vec{q}_2^T \vec{q}_3 & \dots & \vec{q}_2^T \vec{q}_n \\
\vec{q}_3^T \vec{q}_1 & \vec{q}_3^T \vec{q}_2 & \vec{q}_3^T \vec{q}_3 & \dots & \vec{q}_3^T \vec{q}_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vec{q}_n^T \vec{q}_1 & \vec{q}_n^T \vec{q}_2 & \vec{q}_n^T \vec{q}_3 & \dots & \vec{q}_n^T \vec{q}_n
\end{pmatrix} = I
$$

It can be seen, from inverting the order of the factors, that the rows of a square orthonormal matrices are an orthonormal set too.

Complex orthogonal matrices are called “unitary”.