9.58(b)

1 9.58(b), §1 Asked

Asked: Diagonalize

$$q(x,y) = 2x^2 - 6xy + 10y^2$$

2 9.58(b), §2 Solution

$$q = 2x^2 - 6xy + 10y^2$$

Find the matrix of coefficients:

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 10 \end{pmatrix}$$

Eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -3 \\ -3 & 10-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 11 = 0$$

There are two roots: $\lambda_1 = 1$ and $\lambda_2 = 11$

The eigenvector corresponding to λ_1 satisfies

$$\left(\begin{array}{rr|} 1 & -3 \\ -3 & 9 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \end{array} \quad \Longrightarrow \quad \left(\begin{array}{rr|} 1 & -3 \\ 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \\ (2') = (2) + 3(1) \end{array}$$

Taking $v_{1y} = 1$, then $v_{1x} = 3$, giving an eigenvector (3,1). Normalizing this vector to length one gives:

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Big/ \sqrt{3^2 + 1^2} = \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix} = \hat{\imath}'$$

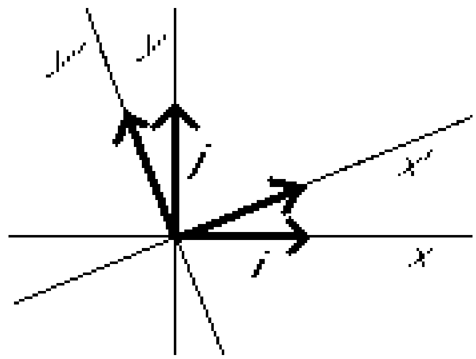
The eigenvector corresponding to λ_2 satisfies

$$\left(\begin{array}{rr|} -9 & -3 \\ -3 & -1 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \end{array} \quad \Longrightarrow \quad \left(\begin{array}{rr|} -9 & -3 \\ 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \\ (2') = 3(2) - (1) \end{array}$$

Taking $v_{2y} = 3$, then $v_{2x} = -1$, giving after normalization:

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \Big/ \sqrt{(-1)^2 + 3^2} = \begin{pmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix} = \hat{\jmath}'$$

Since $1/\sqrt{10} = \sin(18.4°)$, the new axes are rotated 18.5° counter clockwise from the old:



In the new coordinates,

$$q = x'^2 + 11y'^2$$

Note that lines of constant q are now seen to be elliptic.

Important note: It is seen that the quadratic form $\vec{x}^T A\vec{x}$ is always positive for nonzero $\vec{x}$. Symmetric matrices for which this is true are called *positive definite*. They have all positive eigenvalues. Similarly, if all eigenvalues are negative, a symmetric matrix is called *negative definite.* If all eigenvalues are positive or zero, it is called *positive semi-definite.*

Finite element codes for structures typically produce positive definite matrices, as do many other physical applications, such as the kinetic energy of a solid body. Definite matrices are typically easier to deal with in numerical applications than general matrices. For example, no pivoting is needed in the Gaussian elimination involving a definite matrix.