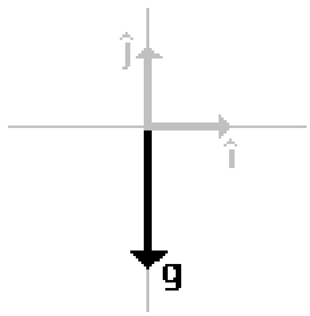
Basis Changes

1 Simple example

Student request: change notations. Mine seem better than the book's, though. I think the books exposition (p207-210) is very confusing, partly by not using vector symbols to indicate vectors versus coordinates. I suggest you stick with my exposition.

To solve problems, it is often desirable or essential to change basis.

As an example, consider the vector of gravity $\vec{g}$. If I use a Cartesian coordinate system $\hat{\imath}, \hat{\jmath}$ with the x-axis horizontal, the vector $\vec{g}$ will be along the negative y-axis. I will call this coordinate system, $(\hat{\imath}, \hat{\jmath})$, the E-system.

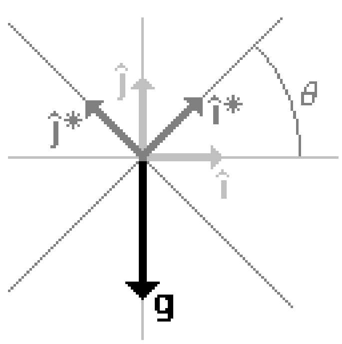


Using the E-system, I can write the vector $\vec{g}$ as:

$$\vec{g} = 0\hat{\imath} - g\hat{\jmath} \quad \text{or } \vec{g}\Big|_E = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

In other words, the *coordinates* of vector $\vec{g}$ in the E-coordinate system are $g_1\Big|_E = 0$ and $g_2\Big|_E = -g$.

But if, say, the ground is under an angle θ with the horizontal, it might be much more convenient to use a coordinate system E^*, $(\hat{\imath}^*, \hat{\jmath}^*)$, with the x-axis aligned with the ground:



In this new coordinate system, the coordinates of $\vec{g}$ will be different. With a bit of trig, you see:

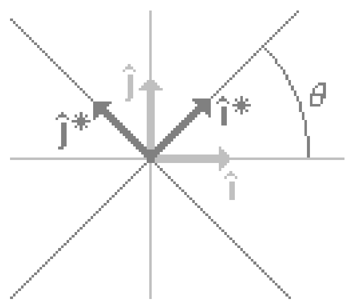
$$\vec{g} = -g\sin(\theta)\hat{\imath}^* - g\cos(\theta)\hat{\jmath}^* \quad \text{or } \vec{g}\Big|_{E^*} = \begin{pmatrix} -g\sin(\theta) \\ -g\cos(\theta) \end{pmatrix}$$

The coordinates of vector $\vec{g}$ are now $g_1\Big|_{E^*} = -g\sin(\theta)$ and $g_2\Big|_{E^*} = -g\cos(\theta)$

What if I need to change the coordinates of a lot of vectors from one coordinate system to the other? Is there a systematic way of doing this? The answer is yes; the following formula applies:

$$\vec{v}\Big|_E = P\vec{v}\Big|_{E^*} \quad \text{with } P = \left(\hat{\imath}^*\Big|_E \; \hat{\jmath}^*\Big|_E\right)$$

So the transformation of coordinates can be done by multiplying by a matrix P. This matrix consists of the basis vectors of the new coordinate system E^* expressed in terms of the old coordinate system E.



In particular,

$$\hat{\imath}^* = \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath} \quad \text{so} \quad \hat{\imath}^*\Big|_E = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

$$\hat{\jmath}^* = -\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath} \quad \text{so} \quad \hat{\jmath}^*\Big|_E = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$$

and matrix P becomes:

$$P = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Let's test it: P times the coordinates of vector $\vec{g}$ in the E^*-system should give the coordinates in the E-system:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} -g\sin(\theta) \\ -g\cos(\theta) \end{pmatrix}$$

Multiplying out gives 0 and $-g$, which is exactly right.

Matrix P is called *the transformation matrix from E to E^*.* Note however that it really transforms coordinates in the E^*-system to coordinates in the E-system. You just have to get used to that language: a transformation matrix from A to B transforms B coordinates into A coordinates. No, I do not know who thought of that first.

What if you really want to transform E coordinates into E^* coordinates? No big deal: just multiply by the inverse matrix P^{-1}.

2 General

The basis vectors do not have to be orthogonal, as in the example. In general, suppose I have a basis S, $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$. Then any arbitrary vector $\vec{w}$ can be written as

$$\vec{w} = w_1\Big|_S \vec{u}_1 + w_2\Big|_S \vec{u}_2 + \ldots + w_n\Big|_S \vec{u}_n$$

where $w_1|_S, w_2|_S, \ldots, w_n|_S$ are the coordinates of $\vec{w}$ in basis S. More briefly,

$$\vec{w}\Big|_S = \begin{pmatrix} w_1\Big|_S \\ \\ w_2\Big|_S \\ \vdots \\ w_n\Big|_S \end{pmatrix}$$

Suppose I have another basis S', $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Then the same vector $\vec{w}$ can also be written as

$$\vec{w} = w_1\Big|_{S'} \vec{v}_1 + w_2\Big|_{S'} \vec{v}_2 + \ldots + w_n\Big|_{S'} \vec{v}_n$$

or

$$\vec{w}\Big|_{S'} = \begin{pmatrix} w_1\Big|_{S'} \\ \\ w_2\Big|_{S'} \\ \vdots \\ w_n\Big|_{S'} \end{pmatrix}$$

The relationship between the two sets of coordinates is always

$$\boxed{\vec{w}\Big|_S = P\,\vec{w}\Big|_{S'}}$$

where P is a matrix that is called the transformation matrix from S to S'. (Although it really works the opposite way.)

Matrix P takes the form:

$$\boxed{P = \left(\vec{v}_1\Big|_S \vec{v}_2\Big|_S \ldots\ \vec{v}_n\Big|_S\right)}$$

It contains the basis vectors of the S' system written in the S system. (That is why if I multiply with P, I get a vector in the S system.)

To get the transformation the other way, use the matrix P^{-1}.